# Idean Extension of E-Bisimple Semigroup Theorems

by

Abdul-Aziz Muhammad Jaser Al-Assaf

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

**MASTER OF SCIENCE**

In

**MATHEMATICS**

September, 1991

Order Number 1354097

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Al-Assaf, Abdul-Aziz Muhammad Jaser Al-Hauwas, M.S.

King Fahd University of Petroleum and Minerals (Saudi Arabia), 1991

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

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# IDEAL EXTENSION OF $E$-BISIMPLE SEMIGROUP THEOREMS

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SEPTEMBER 1991

KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DHAHRAN 31261, SAUDI ARABIA

COLLEGE OF GRADUATE STUDIES

This thesis, written by ABDUL-AZIZ MUHAMMAD JASER AL-HAUWAS AL-ASSAF under the direction of his Thesis Advisor and approved by his Thesis Committee, has been presented to and accepted by the Dean of the College of Graduate Studies, in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE IN MATHEMATICS.

Thesis Committee

R.J. Warne

Dr. R.J. Warne (Chairman)

A. Shafik Ibrahim

Dr. A.S. Ibrahim (Member)

A. Laradji

Dr. A. Laradji (Member)

Dr. Mohammad A. Al-Bar
Department Chairman

Dr. Ala H. Al-Rabeh
Dean, College of Graduate Studies

OCT. 2nd, 1997

Date

**This thesis is dedicated to my parents.**

# خلاصة الرسالة

الاسم : عبدالعزيز محمد جاسر الحواس العساف

عنوان الدراسة : نظريات في التمديد المثالي لشبه الزمرة الإيبايسمبل

التخصص الأساسي: رياضيات

تاريخ الرسالة : سبتمبر ١٩٩١

لقد أعطى البروفيسور وارن النظرية البنائية لشــبه الزمرالإيبايســمبل. باستخدام هذه النظرية ، استطعنا تحديد الترانسليشينال هول لشبه الزمر الإيبايسمبل . وبعد ذلك استخدمنا هذا التحديد لبناء جميع التمديدات المثالية لشبه الزمرة الإيبايسمبل بواسطة شبه الزمرة الزيرو سيمبل المكتملة وأخيرا باستخدام تحديدات البروفيسور وارن لجميع التمديدات المثالية لشبه الزمرة براندت بواسطة شبه زمرة غير محددة ، استطعنا وصف جميع التمديدات المثالية لشبه الزمرة براندت بواسطة شبه الزمرة الإيبايسمبل .

ماجستير في العلوم

جامعة الملك فهد للبترول والمعادن

الظهران ، المملكة العربية السعودية

سبتمبر ١٩٩١م

# ABSTRACT


| | |
|---|---|
| **NAME** | **ABDUL-AZIZ MUHAMMAD JASER AL-HAUWAS AL-ASSAF** |
| **TITLE OF STUDY** | **Ideal Extension of *E*-Bisimple Semigroup Theorems** |
| **MAJOR FIELD** | **Mathematics** |
| **DATE OF DEGREE** | **September 1991** |



Warne gave a structure theorem for $E$-bisimple semigroups. Using this structure theorem, we determine the translational hull of an $E$-bisimple semigroup. We use this determination to construct all ideal extensions of an $E$-bisimple semgroup by a completely 0-simple semigroup. Finally, using Warne's determination of all extensions of a Brandt semigroup by an arbitrary semigroup, we describe all extensions of a Brandt semigroup by an $E$-bisimple semigroup.


**MASTER OF SCIENCE DEGREE**


**KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS**
**Dhahran, Saudi Arabia**


**September 1991**

# ACKNOWLEDGEMENTS

Praise be to Allah, Lord of the Worlds, the Almighty, with Whose gracious help it was possible to accomplish this task.

I have the greatest pleasure in acknowledging my gratitude to my Thesis Advisor, Dr. Ronson J. Warne, who introduced me to the subject and for his patient guidance, moral support and continuous encouragement. I would also like to express my gratitude to the other Thesis Committee members, Dr. A. Laradji and Dr. A.S. Ibrahim who took the trouble of going through the manuscript and giving some very helpful suggestions and remarks.

Grateful appreciation must be expressed to Dr. Mohammad Al-Bar, the Chairman of the Department, in acknowledgement of his most generous help and cooperation.

Thanks are due to my brothers and friends for their encouragement. Special thanks are due to Walid Al-Jasem for his assistance in producing this work in a nice. form

# INTRODUCTION

Let $E$ be a band (idempotent semigroup). The collecton $E(\mathcal{R})$ of $\mathcal{R}$-classes of $E$ may be partially ordered by the following rule. If $R_1, R_2 \in E(\mathcal{R})$, $R_1 < R_2$ if and only if $e < f$ for all $e \in R_1$ and $f \in R_2$ ($e \leq f$ if and only if $ef = fe = e$). If $E(\mathcal{R})$, under this order, is order isomorphic to $I^0$, the non-negative integers, under the reverse of the usual order, $E$ is called a naturally ordered band. A bisimple semigroup whose idempotents form a naturally ordered band is termed an $E$-bisimple semigroup. In [27], Warne showed that $S$ is an $E$-bisimple semigroup if and only if $S = (I^0 \times \{0\}) \times (G \times P)) \cup ((I^0 \times N) \times (G \times K))$, where $G$ is a group, $N$ is the natural numbers, and $P$ and $K$ are sets under the multiplication $((n,k),(g,p))((r,s),(h,q)) = ((n+r-t, k+s-t),(g\theta^{r-t}h\theta^{k-t}, x))$ where juxtaposition denotes multiplication in $G, \theta$ is an endomorphism of $G$($\theta^0$ is the identity automorphism), $t = \min(r,k)$, $x = q$ or $p(h\theta^{k-r-1}\gamma)$. According to whether $r \geq k$ or $k > r$ and $\gamma$ is a homomorphism of $G$ into $G_K$, the symmetric group on $K$.

In chapter 1 of this thesis, we give basic definitions, concepts and theorems of the semigroup theory which can be found in [7, 8] and which we use in the succeeding two chapters. We also establish two new lemmas (lemma 1.3 and 1.13) which we use in the sequel. Furthermore, we state the concept of ideal extension given by a partial homomorphism due to Warne [15]. We also give the structure theorem for $E$-bisimple semigroups due to Warne [27] (Theorem 1.11) which is mentioned above and the determination of Green's relations for $E$-bisimple semigroup also due to Warne [27] (Lemma 1.12). We give the extensions of a Brandt semigroup by an arbitrary semigroup with zero (Theorem 1.16) due to Warne [16, 18]. By the way, there are more than one kind of extensions of semigroups but in this thesis we deal with ideal extension only and if we say "extension" any where in this thesis we mean "ideal extension".

In chapter 2, we present right and left translations of an $E$-bisimple semigroup $S$ and then we give a sufficient condition for linkage. After that we establish some lemmas to give a more clear picture and useful form of right translations and we present a relation between left translations. Finally, we establish the main result of this chapter (Theorem 2.7) which gives the structure of the translational hull $\bar{S}$ of the $E$-bisimple semigroup $S$.

In chapter 3, we establish two main results. The first one is the determination of all extensions of an $E$-bisimple semigroup by a completely 0-simple semigroup (which is given by Theorem 3.2) and the second one is the determination of all extensions of a Brandt semigroup by an $E$-bisimple semigroup (which is given by Theorem 3.4). Let $S$ be an $E$-bisimple semigroup, let $T$ be a completely 0-simple semigroup, let $T^* = T\backslash 0$ and let $H$ be a group. Theorem 3.1 determines the fashion of any partial homomorphism of $T^*$ into $S$ and it is used in Theorem 3.2. Theorem 3.3 determines the fashion of any homomorphism of $S$ into $H$ and it is used in Theorem 3.4. However, all of the results (Lemmas and Theorems) of chapter 2 and 3 of this thesis are new.

If $S$ is a semigroup, $E(S)$ will denote the set of idempotents of $S$. If $\alpha$ is an order type, $\alpha^*$ denotes with the convese order. We term $S$ an $\alpha$-semigroup if $E(S)$ with its usual order has order type $\alpha^*$. The structure of $\omega$-bisimple inverse semigroups was given by Reilly [13] and Warne [17]. The structure of $\omega^n$-bisimple inverse semigroups was given by Warne [20]. The structure of $I$-bisimple inverse semigroups and $\omega^n$ $I$-bisimple inverse semigroups was given by Warne in [21] and [26] respectively. The structure of $\omega$-inverse semigroups was given by Munn [9] and the structure of $I$-inverse semigroups was given by warne [24]. For another approach to structure theory see [28]. Various properties of these semigroups (i.e. the determination of homomorphisms, congruences, ideal extensions, study of the lattice of congruences) have been investigated, for example, by Baird [1, 2], Bonzini and Cherubini [3–5], Munn [10], Munn and Reilly [11], Petrich [12], Scheiblich [14], and WArne [19, 21–23, 25, 26.]

Unlike the semigroups in the paragraph above, the $E$-bisimple semigroups are not inverse semigroups nor are they $\mathcal{H}$-compatible. Nevertheless, the structure theorem is of sufficient simplicity to yield a homomorphism theory and an ideal extension theory. A determination of the congruence relations and an investigation of the lattice of congruences will be the subject of future papers.

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# CHAPTER 1
# PRELIMINARIES

In this chapter we introduce a minimal amount of basic concepts of semigroup theory, just enough to serve as a working tool for the remainder of this thesis. We introduce the basic definitions of semigroup theory and establish the basic terminology and notation. We also discuss the Green's relations and some classes of semigroups which are needed later. Finally, we present some needed theorems and definitions of ideal extension theory.

## Basic Definitions

**Definition 1.1** *A mapping $f$ from a set $X$ to a set $Y$ (written $f : X \to Y$) is a correspondence that assigns to each element of $X$ exactly one element of $Y$. $f$ is said to be onto if every element of $Y$ is the image under $f$ of at least one element of $X$. $f$ is said to be $1-1$ if distinct elements of $X$ are mapped by $f$ into distinct elements of $Y$.*

**Definition 1.2** *A binary operation on a set $S$ is a mapping of $S \times S$ into $S$ where $S \times S$ is the set of all ordered pairs of elements of $S$. A groupoid is a system $(S, \cdot)$*

*consisting of a non-empty set $S$ together with a binary operation "." on $S$. We shall usually write $S$ instead of $(S, .)$ when there is no danger of ambiguity.*

**Definition 1.3** *A partial binary operation on a set $S$ is a mapping of a non-empty subset of $S \times S$ into $S$. By a partial groupoid we shall mean a system $(S, .)$ consisting of a non-empty set $S$ together with a partial binary operation "." on $S$.*

**Definition 1.4** *A binary operation "." on a set $S$ is called associative if $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c$ in $S$. A semigroup is a groupoid $(S, \cdot)$ such that the operation "." is associative. A non-empty subset $T$ of a semigroup $S$ is called a subsemigroup of $S$ if $a \in T$ and $b \in T$ imply $a \cdot b \in T$. Frequently we shall omit the dot, writing $ab$ for $a \cdot b$.*

**Definition 1.5** *A $1-1$ mapping of a set $X$ upon itself will be called a permutation of $X$, even when $X$ is infinite. The full symmetric group $G_X$ on $X$ consists of all permutations of $X$ under the operation of iteration.*

**Definition 1.6** *By an ideal of a groupoid $S$ we mean a non-empty subset $A$ of $S$ such that $SA \subseteq A$ and $AS \subseteq A$. A groupoid $S$ is called simple if it contains no proper ideal.*

**Definition 1.7** *Let $S$ be a semigroup and let $a$ and $b$ be in $S$. If $ax = bx$ and $xa = xb$ for all $x \in S$ imply $a = b$, the semigroup $S$ is said to be weakly reductive.*

**Definition 1.8** *If $\varphi$ is a mapping from a semigroup $S$ into a semigroup $T$, we say that $\varphi$ is a homomorphism if, for all $x, y \in S$, $(xy)\ \varphi = (x\varphi)(y\varphi)$. If $\varphi$ is $1-1$, we shall call it a monomorphism, and if it is both $1-1$ and onto we shall call it an isomorphism. If $\varphi$ is a homomorphism from $S$ into $S$ we call it an endomorphism of $S$. An isomorphism from $S$ onto $S$ will be called automorphism of $S$. If there exists an isomorphism $\varphi$ from $S$ onto $T$, we say that $S$ and $T$ are isomorphic. If $S$ and $T$ are groups, we define kernel of $\varphi$ (written ker $\varphi$) to be the set of all elements $x$ in $S$ such that $x\varphi = e_T$.*

**Definition 1.9** *By a partial homomorphism of a partial groupoid $S$ into a partial groupoid $T$, we mean a mapping $\varphi$ of $S$ into $T$ such that if $x$ and $y$ are elements of $S$ such that $xy$ is defined in $S$, then the product $(x\varphi)(y\varphi)$ is defined in $T$ and is equal to $(xy)\varphi$.*

**Definition 1.10** *An element $e$ of a groupoid $S$ is called idempotent if $ee = e$.*

**Lemma 1.1** *Let $\varphi$ be a [partial] homomorphism of a semigroup $S$ into a semigroup $T$. Then $\varphi$ maps idempotent elements of $S$ into idempotent elements of $T$.*

**Definition 1.11** *An element $a$ of a semigroup $S$ is called regular if $a \in aSa$, that is, there exists an element $x$ in $S$ such that $a = axa$. A semigroup $S$ is called regular if every element of $S$ is regular.*

**Definition 1.12** *Two elements $a$ and $b$ of a semigroup $S$ are said to be inverses of each other if $aba = a$ and $bab = b$. By an inverse semigroup we mean a semigroup in which every element has a unique inverse.*

**Theorem 1.2** *The following three conditions on a semigroup $S$ are equivalent.*

*i) $S$ is regular, and any two idempotent elements of $S$ commute with each other;*

*ii) every principal right ideal and every principal left ideal of $S$ has a unique idempotent genrator;*

*iii) $S$ is an inverse semigroup.*

**Definition 1.13** *By a $1-1$ partial transformation of a set $X$ we mean a $1-1$ mapping $\alpha$ of a subset $Y$ of $X$ upon a subset $Y' = Y\alpha$ of $X$. By the inverse $\alpha^{-1}$ of $\alpha$ we mean the mapping of $Y\alpha$ upon $Y$ which is inverse to $\alpha$ in the usual sense of mapping, i.e., $y'\alpha^{-1} = y (y \in Y, y' \in Y\alpha)$ if and only if $y' = y\alpha$.*

**Definition 1.14** *Let $\mathcal{I}_X$ denote the set of all $1-1$ partial transformations of $X$, including that of the empty subset $\Box$ upon itself, this "empty transformation" will be denoted by $0$. The product $\alpha\beta$ of two elements $\alpha$ and $\beta$ of $\mathcal{I}_X$ is defined as follows. Let $Y$ be the domain of $\alpha$ and $Z$ that of $\beta$. If $Y\alpha \cap Z = \Box$, we define $\alpha\beta = 0$. Otherwise, let $W = (Y\alpha \cap Z)\alpha^{-1}$. Then we define $\alpha\beta$ to be the iterate of $\alpha_{|W}$ and*

*$\beta_{|W\alpha}$ in the usual sense. Clearly, $\alpha\beta$ is $1-1$ transformation of $W$ upon $W\alpha\beta$ and so belongs to $\mathcal{I}_X$. Associativity is easily verified. Hence $\mathcal{I}_X$ is a semigroup, which we call the symmetric inverse semigroup on the set $X$.*

## Green's Relations

Green's relations are very useful in many areas of semigroup theory. For example, the structure of semigroups in general became much more feasible after using Green's equivalence relations. We will use them in the constructions of some of our proofs in chapter 3.

Let $S$ be a semigroup, we define $S^1$ to be $S$ if $S$ has an identity element and otherwise to be $S$ with an identity element 1 adjoined.

**Definition 1.15** *Two elements a and b of a semigroup S are said to be $\mathcal{R}$- equivalent i.e., $a\mathcal{R}b$ if and only if $aS^1 = bS^1$ and this is the case if and only if there exist $x, y \in S^1$ such that $ax = b$ and $a = by$.*

*So, $\mathcal{R} = \{(a,b) | aS^1 = bS^1\} \subseteq S \times S$.*

*Let $R_a = \{x | x\mathcal{R}a\}$, $R_a$ is called the $\mathcal{R}$-class containing a.*

**Definition 1.16** *Two elements a and b of a semigroup $S$ are said to be $\mathcal{L}$-equivalent i.e., $a\mathcal{L}b$ if and only if $S^1a = S^1b$ and this is the case if and only if there exist $x, y \in S^1$ such that $xa = b$ and $a = yb$.*

So, the relation $\mathcal{L} = \{(a,b) | S^1a = S^1b\} \subseteq S \times S$.

Let $L_a = \{x | x\mathcal{L}a\}$, $L_a$ is called the $\mathcal{L}$-class containing $a$.

**Definition 1.17** *Two elements $a$ and $b$ of a semigroup $S$ are said to be $\mathcal{H}$- equivalent if and only if they are $\mathcal{R}$-equivalent and $\mathcal{L}$- equivalent, i.e., $a\mathcal{H}b$ iff $a\mathcal{R}b$ and $a\mathcal{L}b$.*

So the relation $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$.

Let $H_a = \{x | x\mathcal{H}a\}$, $H_a$ is called the $\mathcal{H}$-class containing $a$ and equals to $R_a \cap L_a$.

**Definition 1.18** *Two elements $a$ and $b$ of a semigroup $S$ are said to be $\mathcal{D}$-equivalent if and only if there exists an element $c \in S$ such that $a, c$ are $\mathcal{R}$-equivalent and $c, b$ are $\mathcal{L}$-equivalent, i.e., $a\mathcal{D}b$ iff $\exists c \in S$ such that $a\mathcal{R}c$ and $c\mathcal{L}b$. We may also say that $a\mathcal{D}b \Leftrightarrow R_a \cap L_b \neq \square \Leftrightarrow L_a \cap R_b \neq \square$.*

Let $D_a = \{x | x\mathcal{D}a\}$, $D_a$ is called the $\mathcal{D}$-class containing $a$.

The relation $\mathcal{D} = \mathcal{R}o\mathcal{L} = \mathcal{L}o\mathcal{R}$ is the smallest equivalent relation containing both $\mathcal{L}$ and $\mathcal{R}$.

We say that $S$ is $\mathcal{D}$-simple or bisimple if it consists of a single $\mathcal{D}$-class.

**Definition 1.19** *Two elements $a$ and $b$ of a semigroup $S$ are said to be $\mathcal{I}$-equivalent if and only if $S^1aS^1 = S^1bS^1$ and this is the case if and only if there exist $x, y, u, v \in S^1$ such that $xay = b$ and $ubv = a$.*

So, the relation $\mathcal{I} = \{(a,b) | S^1aS^1 = S^1bS^1\} \subseteq S \times S$.

*Let $J_a = \{x | x\mathcal{I}a\}$, $J_a$ is called the $\mathcal{I}$-class containing $a$.*

*It is very clear that $\mathcal{L} \subseteq \mathcal{I}$ and $\mathcal{R} \subseteq \mathcal{I}$ which imply that $\mathcal{D} \subseteq \mathcal{I}$. Note that a semigroup $S$ is simple if and only if it consists of a single $\mathcal{I}$-class. So every bisimple semigroup is also simple.*

**Lemma 1.3** *Let $\varphi$ be a partial homomorphism of a semigroup $S$ into a semigroup $T$. Then, $\varphi$ maps $\mathcal{R}$-classes, $\mathcal{L}$-classes and $\mathcal{D}$-classes of $S$ into $\mathcal{R}$-classes, $\mathcal{L}$-classes and $\mathcal{D}$-classes of $T$ respectively.*

Proof: First, we want to show that $\varphi$ maps $\mathcal{R}$-classes of $S$ into $\mathcal{R}$-classes of $T$. Let $R_a$ be the $\mathcal{R}$-class of $S$ containing $a$ and let $R_{a\varphi}$ be the $\mathcal{R}$-class of $T$ containing $a\varphi$. Let $b \in R_a$ i.e., $a\mathcal{R}b$ where $a, b \in S$. So, $\exists x, y \in S^1$ such that $ax = b$ and $a = by$. Thus $a\varphi = (by)\varphi = (b\varphi)(y\varphi)$ and $b\varphi = (ax)\varphi = (a\varphi)(x\varphi)$. Hence for $a\varphi, b\varphi \in T$, $\exists x\varphi, y\varphi \in T^1$ such that $a\varphi = (b\varphi)(y\varphi)$ and $b\varphi = (a\varphi)(x\varphi)$. Therefore $a\varphi\mathcal{R}b\varphi$ which implies that $b\varphi \in R_{a\varphi}$.

Similarly, we can show that $\varphi$ maps $\mathcal{L}$-classes of $S$ into $\mathcal{L}$-classes of $T$.

Finally, we want to show that $\varphi$ maps $\mathcal{D}$-classes of $S$ into $\mathcal{D}$-classes of $T$. Let $D_a$ be the $\mathcal{D}$-class of $S$ containing $a$ and let $D_{a\varphi}$ be the $\mathcal{D}$-class of $T$ containing $a\varphi$. Let $b \in D_a$ i.e., $a\mathcal{D}b$ where $a, b \in S$. So, $\exists c \in S$ such that $a\mathcal{R}c$ and $c\mathcal{L}b$. Hence from the above parts of the proof we have $a\varphi\mathcal{R}c\varphi$ and $c\varphi\mathcal{L}b\varphi$. Therefore, $a\varphi\mathcal{D}b\varphi$ which implies that $b\varphi \in D_{a\varphi}$.

## Completely [0-]Simple Semigroups

**Definition 1.20** *An element $z$ of a semigroup $S$ is called a zero element if $zx = xz = z$ for every $x \in S$. A semigroup $S$ with zero element 0 is called 0-simple if (i) $S^2 \neq 0$, (ii) 0 is the only proper ideal of $S$.*

**Definition 1.21** *Let $E$ be the set of idempotents of a semigroup $S$. If $e, f \in E$, we define $e \leq f$ to mean $ef = fe = e$. It is well known that $\leq$ is a partial ordering of $E$. If $S$ contains a zero element 0, then $0 \leq e \quad \forall e \in E$. An idempotent element $f \in S$ is called primitive if $f \neq 0$ and if $e \leq f$ implies $e = 0$ or $e = f$. By a completely [0-]simple semigroup we mean a [0-]simple semigroup containing a primitive idempotent.*

If $H$ is a group then $H^0 = H \cup \{0\}$ is a semigroup; it is not of course a group; we call it a 0-group.

**Theorem 1.4** *(Rees's Theorem) Let $H^0$ be a 0-group, let $F, \Lambda$ be non-empty sets, and let $Q = (q_{mi})$ be a $\Lambda \times F$ matrix with entries in $H^0$. Suppose that $Q$ is regular in the sense that no row or column consists entirely of zeros: formally, $\forall i \in F$, $\exists m \in \Lambda$ such that $q_{mi} \neq 0$ and $\forall m \in \Lambda$, $\exists i \in F$ such that $q_{mi} \neq 0$. Let $S = (H \times F \times \Lambda) \cup \{0\}$ and define a binary operation on $S$ by*

$$(a, i, m)(b, j, n) = \begin{cases} (aq_{mj}b, i, n) & \text{if} \quad q_{mj} \neq 0 \\ 0 & \text{if} \quad q_{mj} = 0 \end{cases}$$

$$(a,i,m)0 = 0(a,i,m) = 00 = 0.$$

*Then $S$ is a completely 0-simple semigroup.*

*Conversely, every completely 0-simple semigroup is isomorphic to one constructed in this way.*

*$S$ will be denoted by $M^0(H,F,\Lambda,Q)$ and will be called Rees $F \times \Lambda$ matrix semigroup over the 0-group $H^0$ with the regular sandwich matrix $Q$.*

**Lemma 1.5** *Two Rees $F \times \Lambda$ matrix semigroups $S = M^0(H,F,\Lambda,Q)$ and $S' = M^0(H,F,\Lambda,Q')$ over the same group with zero $H^0$ are isomorphic if there exist a mapping $i \rightarrow u_i$ of $F$ into $H$ and a mapping $m \rightarrow v_m$ of $\Lambda$ into $H$ such that $q'_{mi} = v_m q_{mi} u_i$ for all $i \in F$ and $m \in \Lambda$, where $Q = (q_{mi})$ and $Q' = (q'_{mi})$.*

**Lemma 1.6** *Let $S = M^0(H,F,\Lambda,Q)$ be a completely 0-simple semigroup. Then we may assume that $1 \in F \cap \Lambda$, $q_{1i}$ is either 0 or $e_H$ for all $i \in F$ and $q_{m1}$ is either 0 or $e_H$ for all $m \in \Lambda$.*

**Theorem 1.7** *Let $H$ be a group, let $F, \Lambda$ be non-empty sets, and let $Q = (q_{mi})$ be a $\Lambda \times F$ matrix with entries in $G$. Let $S = H \times F \times \Lambda$ and define a binary operation on $S$ by*

$$(a,i,m)(b,j,n) = (aq_{mj}b, i, n).$$

*Thus, $S$ is a completely simple semigroup.*

*Conversely, any completely simple semigroup is isomorphic to one constructed in this manner.*

$S$ will be denoted by $M(H,F,\Lambda,Q)$ and will be called the Rees $F \times \Lambda$ matrix semigroup without zero over the group $H$ with sandwich matrix $Q$.

**Lemma 1.8** *Let $S = M(H,F,\Lambda,Q)$ be a completely simple semigroup. Then we may assume that $1 \in F \cap \Lambda$ and $q_{1i} = q_{m1} = e_H$ for all $i \in F$ and all $m \in \Lambda$.*

**Lemma 1.9** *Let $S = M^0(H,F,\Lambda,Q)$ be a completely 0-simple semigrouip. Then*

$$(a,i,m)\mathcal{L}(b,j,n) \text{ if and only if } m = n,$$
$$(a,i,m)\mathcal{R}(b,j,n) \text{ if and only if } i = j.$$

*It follows now that the set of non-zero $\mathcal{R}$-classes in $S$ is $\{R_{i_0} = R_{(a,i_0,m)} | i_0 \in F\}$ where*

$$R_{i_0} = \{(a,i_0,m) | a \in H,\ m \in \Lambda\}$$

*and the set of non-zero $\mathcal{L}$-classes in $S$ is $\{L_{m_0} = L_{(a,i,m_0)} | m_0 \in \Lambda\}$ where*

$$L_{m_0} = \{(a,i,m_0) | a \in H,\ i \in F\}.$$

## Brandt Semigroups

**Definition 1.22** *A Brandt semigroup is a semigroup $S$ with zero satisfying the following axioms:*

*(i) If $a, b, c$ are elements of $S$ such that $ac = bc \neq 0$ or $ca = cb \neq 0$, then $a = b$.*

*(ii) If $a, b, c$ are elements of $S$ such that $ab \neq 0$ and $bc \neq 0$, then $abc \neq 0$.*

*(iii) To each element $a \neq 0$ of $S$ there correspond a unique element $e$ of $S$ such that $ea = a$, a unique element $f$ of $S$ such that $af = a$, and a unique element $a'$ of $S$ such that $a'a = f$.*

*(iv) If $e$ and $f$ are nonzero idempotents of $S$ then $eSf \neq 0$.*

Actually, Brandt semigroups are special class of completely 0-simple semigroups as we may see from the following theorem.

**Theorem 1.10** *The following three conditions on a semigroup $S$ with zero are equivalent.*

*i) $S$ is a Brandt semigroup.*

*ii) $S$ is a completely 0-simple inverse semigroup.*

*iii) $S$ is isomorphic with a (regular) Rees $I \times I$ matrix semigroup $M^0(G, I, I, \Delta)$ over a group with zero $G^0$ and with the $I \times I$ identity matrix $\Delta$ as sandwich matrix.*

## $E$-Bisimple Semigroups

**Definition 1.23** *Let $E$ be an idempotent semigroup (a band). The collection $E(\mathcal{R})$ of $\mathcal{R}$-classes of $E$ may be partially ordered by the following rule. If $R_1, R_2 \in$*

*$E(\mathcal{R})$, $R_1 < R_2$ if and only if $e < f$ for all $e \in R_1$ and $f \in R_2$. ($e \leq f$ if and only if $ef = fe = e$). If $E(\mathcal{R})$, under this order is order isomorphic to $I^0$, the non-negative integers, under the reverse of the usual order, $E$ is called a naturally ordered band. A bisimple semigroup whose idempotents form a naturally ordered band is termed an E-bisimple semigroup.*

$E$-bisimple semigroups can be characterized by the following theorem which is due to Professor Warne and it gives the structure of any $E$-bisimple semigroup.

**Theorem 1.11** *$S$ is an E-bisimple semigroup if and only if*

$$S \cong ((I^0 \times \{0\}) \times (G \times P)) \cup ((I^0 \times N) \times (G \times K))$$

*where $I^0$ is the set of nonnegative integers, $N$ is the set of natural numbers, $G$ is a group and $P$ &$K$ are sets, under the multiplication*

$$((n,k),(g,p))((r,s),(h,q)) = \begin{cases} ((n+r-k,s),(g\theta^{r-k}h,q)) & \text{if } r \geq k \\ ((n,k+s-r),(g(h\theta^{k-r}),p(h\theta^{k-r-1}\gamma))) & \text{if } k > r \end{cases}$$

*where $\theta$ is an endomorphism on $G$ and $\gamma$ is a homomorphism of $G$ into $G_K$, the symmetric group on $K$.*

**Lemma 1.12** *Let $S = (G,P,K,\theta,\gamma)$ be an E-bisimple semigroup. Then*

*(a)* $((n,k),(g,p))\mathcal{R}((r,s),(h,q))$ *if and only if* $n = r$

*(b)* $((n,k),(g,p))\mathcal{L}((r,s),(h,q))$ *if and only if* $k = s$ *and* $p = q$

*(c)* $E_S = \{((0,0),(e,p)) | p \in P\} \cup \{((n,n),(e,q)) | q \in K\}$ *where* $E_S$ *is the set of idempotent elements of* $S$ *and* $e$ *is the identity element of the structure group* $G$.

So, it follows that the set of $\mathcal{L}$-classes in an $E$-bisimple semigroup $S$ is $\{L_{k_1,q_1} = L_{((n_1,k_1),(g_1,q_1))} |$ either $k_1 = 0$, $q_1 \in P$ or $k_1 \in N$, $q_1 \in K\}$ where

$$L_{k_1,q_1} = \{((n,k_1),(g,q_1)) | n \in I^0,\ g \in G\},$$

and the set of $\mathcal{R}$-classes in $S$ is $\{R_{n_2} = R_{((n_2,k_2),(g_2,q_2))} | n_2 \in I^0\}$ where

$$R_{n_2} = \{((n_2,k),(g,q)) | g \in G \text{ and either } k = 0, q \in P \text{ or } k \in N,\ q \in K\}.$$

However, an $E$-bisimple semigroup consists of a single $\mathcal{D}$-class since it is a bisimple semigroup.

**Lemma 1.13** *Let* $S$ *be an* $E$*-bisimple semigroup. Then* $S$ *is a weakly reductive semigroup.*

<u>Proof</u>: Let $((n,k),(g,p)), ((r,s),(h,q)) \in S$ and suppose that for all $((a,b),(x,y)) \in S$ we have

$$((n,k),(g,p))((a,b),(x,y)) = ((r,s),(h,q))((a,b),(x,y)) \quad (*)$$
$$\text{and} \quad ((a,b),(x,y))((n,k),(g,p)) = ((a,b),(x,y))((r,s),(h,q)) \quad (**)$$

So, in particular, it is true for $((0,0),(e,z)) \in S$ where $z \in P$. Hence, from (**), we get

$$((n,k),(g,p)) = ((r,s),(h,q)).$$

Therefore, $S$ is weakly reductive.

## The Translational Hull of a Semigroup

**Definition 1.24** *A transformation $\rho$ of a semigroup $S$ is called a right translation of $S$ if $x(y\rho) = (xy)\rho \ \forall x, y \in S$.*

**Definition 1.25** *A transformation $\rho_a$ of a semigroup $S$ is called the inner right translation of $S$ corresponding to the element $a$ of $S$ if $x\rho_a = xa \ \forall x \in S$.*

**Definition 1.26** *A transformation $\lambda$ of a semigroup $S$ is called a left translation of $S$ if $(\lambda x)y = \lambda(xy) \ \forall x, y \in S$.*

**Definition 1.27** *A transformation $\lambda_a$ of a semigroup $S$ is called the inner left translation of $S$ corresponding to the element $a$ of $S$ if $\lambda_a x = ax, \ \forall x \in S$.*

**Definition 1.28** *A right translation $\rho$ and a left translation $\lambda$ are said to be linked if $x(\lambda y) = (x\rho)y \ \forall x, y \in S$.*

**Definition 1.29** *We define the translational hull $\bar{S}$ of a semigroup $S$ to be the set of all pairs $(\rho, \lambda)$ of linked right and left translations $\rho$ and $\lambda$ of $S$. It is well known that $\bar{S}$ is a semigroup.*

## Ideal Extension Theory

**Definition 1.30** *Let $S$ be an ideal of a semigroup $V$. For all $a, b \in V$, define $a\delta b$ to mean that either $a = b$ or else both $a, b \in S$. We call $\delta$ the Rees congruence modulo $S$. The equivalence classes of $V$ mod $\delta$ are $S$ itself and every one element set $\{a\}$ with $a \in V\backslash S$. We shall write $V/S$ instead of $V/\delta$, and we call $V/S$ the Rees factor semigroup of $V$ modulo $S$.*

**Definition 1.31** *Let $S$ and $T$ be disjoint semigroups, $T$ having a zero element $0$. A semigroup $V$ will be called an ideal extension of $S$ by $T$ if it contains $S$ as an ideal, and if the Rees factor semigroup $V/S$ is isomorphic with $T$.*

**Definition 1.32** *If $V$ is an ideal extension of a semigroup $S$ by a semigroup $T$ with zero, then we say that $V$ is given by a partial homomorphism if there exists a partial homomorphism $\pi : T^* \to S$ where $T^* = T\backslash 0$ such that*

$$AoB \begin{cases} AB & \text{if } AB \neq 0 \text{ (in } T) \\ (A\pi)(B\pi) & \text{if } AB = 0 \text{ (in } T) \end{cases}$$

$$Aoc = (A\pi)c, \; coA = c(A\pi), \; cod = cd$$

*where $A, B \in T^*$, $c, d \in S$, $o$ denotes the multiplication in $V$ and juxtaposition denotes multiplication in $S$ and $T$.*

**Theorem 1.14** *A partial homomorphism $A \to \bar{A}$ of the partial groupoid $T^*$ into $S$ determines an extension $(V, o)$ of $S$ by $T$ as follows:*

$$AoB = \begin{cases} AB & \text{if } AB \neq 0 \\ \bar{A}\bar{B} & \text{if } AB = 0; \end{cases}$$

$$Aos = \bar{A}s,\ soA = s\bar{A};\ sot = st.$$

*If $S$ has an identity element, then every extension of $S$ by $T$ is found in this fashion.*

**Theorem 1.15** *Let $S$ be a weakly reductive semigroup, and let $\bar{S}$ be its translational hull. Let $T$ be a semigroup with zero 0, and let $T^* = T \backslash 0$. Let $V = T^* \cup S$ and $\bar{V} = T^* \cup \bar{S}$. Let $(\bar{V}, o)$ be an extension of $\bar{S}$ by $T$. Then $(V, o)$ is an extension of $S$ by $T$ if and only if $(V, o)$ is a subsemigroup of $(\bar{V}, o)$, and this is the case if and only if $AoB \in S$ for every pair of elements $A, B$ of $T$ such that $AB = 0$ in $T$.*

*Conversely, let $(V, o)$ be an extension of $S$ by $T$. Then there exists an extension $(\bar{V}, o)$ of $\bar{S}$ by $T$ such that $(V, o)$ is a subsemigroup of $(\bar{V}, o)$.*

**Theorem 1.16** *Let $V$ be an extension of a Brandt semigroup $S$ by an arbitrary semigroup $T$ with zero. Let $S$ be given the Rees representation $S = M^0(H, I, I, \Delta)$. Then, there exists a partial homomorphism $W : T^* \to \mathcal{I}_I$, the full symmetric inverse semigroup on $I$ such that $AW = W_A$. Let $s_A$ and $t_A$ denote the domain and the range of $W_A$ respectively and let 0 and 0′ denote the zeros of $S$ and $T$ respectively. If $AB = 0'$, either $t_A \cap s_B = \square$ or $t_A \cap s_B$ is a single element $d_{A,B}$. For each $A \in T^*$,* *there exists a mapping $\psi_A$ of $s_A$ into the group $H$ such that for $AB \neq 0'$*

$$(i\psi_A)(iW_A\psi_B) = i\psi_{AB} \quad \text{for all} \quad i \in s_{AB} \qquad (*)$$

*The products in $V$ are given by*

$$AoB = \begin{cases} AB & \text{if} \quad AB \neq 0' \\ 0 & \text{if} \quad AB = 0' \text{ and } t_A \cap s_B = \square \\ ((d_{A,B}W_A^{-1}\psi_A)(d_{A,B}\psi_B), d_{A,B}W_A^{-1}, d_{A,B}W_B) & \text{if} \quad AB = 0' \text{ and } t_A \cap s_B = d_{A,B} \end{cases} \tag{1}$$

$$(a,i,m)oA = \begin{cases} (a(m\psi_A), i, mW_A) & \text{if} \quad m \in s_A \\ 0 & \text{if} \quad m \notin s_A \end{cases} \tag{2}$$

$$Ao(a,i,m) = \begin{cases} ((iW_A^{-1}\psi_A)a, iW_A^{-1}, m) & \text{if } i \in t_A \\ 0 & \text{if } i \notin t_A \end{cases} \tag{3}$$

$$0oA = Ao0 = 0. \tag{4}$$

*Conversely, let $S$ be a Brandt semigroup and $T$ be a semigroup with zero such that $S \cap T = \square$. If we are given the mappings $W$ and $\psi_A$ described above and define product $o$ in the class sum of $S$ and $T^*$ by (1) – (4), then $V$ is an extension of $S$ by $T$.*

**Theorem 1.17** *Let $S = M^0(H, F, \Lambda, Q)$ be a completely 0-simple semigroup and let $G$ be a group. Let $i \to u_i$ and $m \to v_m$ be mnappings of $F$ into $G$ and $\Lambda$ into $G$ respectively and let $\omega$ be a homomorphism of $H$ into $G$ such that $q_{mi}\omega = v_m u_i$*

*if* $q_{mi} \neq 0$. *Then* $(a, i, m)\varphi = u_i a \omega v_m$ *defines a partial homomorphism of* $T^*$ *(i.e.,* $T\backslash 0$) *into* $G$, *and every partial homomorphism of* $T^*$ *into* $G$ *is obtained in this way.*

# CHAPTER 2

# THE TRANSLATIONAL HULL OF AN $E$-BISIMPLE SEMIGROUP

The translational hull of a semigroup $S$ plays a very important role in finding all possible extensions of $S$ by any other semigroup $T$ with zero as you will see later on in Chapter 3.

In this chapter we construct the translational hull of any $E$-bisimple semigroup $S$ and to do that, first of all, we present the right and left translations and then we give a sufficient condition for linkage. After that, we give some lemmas to help us to prove the theorem which determines the multiplication of the translational hull.

**Lemma 2.1** *$\rho_\phi$ is a right translation of an E-bisimple semigroup S if and only if*

$$((n,k),(g,p))\rho_\phi = \begin{cases} ((n,k),(g,p))\phi(z) & \text{if } k>0 \\ ((n,0),(g,p))\phi(p) & \text{if } k=0 \end{cases}$$

*where $z$ is any element in $P$ and $\phi$ is a mapping from $P$ into $S$.*

<u>Proof</u>: ($\Rightarrow$) let $\rho$ be any right translation of an $E$-bisimple semigroup $S$. Since for all $((n,k),(g,p)) \in S$

$$((n,k),(g,p)) = \begin{cases} ((n,k),(g,p))((0,0),(e,q)) & \text{if } k>0 \\ ((n,0),(g,p))((0,0),(e,p)) & \text{if } k=0 \end{cases}$$

where $q$ is any element in $P$. Then,

$$((n,k),(g,p))\rho \;=\; \begin{cases} [((n,k),(g,p))((0,0),(e,q))]\rho & \text{if } k>0 \\ [((n,0),(g,p))((0,0),(e,p))]\rho & \text{if } k=0 \end{cases}$$

$$\;=\; \begin{cases} ((n,k),(g,p))[((0,0),(e,q))\rho] & \text{if } k>0 \\ ((n,0),(g,p))[((0,0),(e,p))\rho] & \text{if } k=0 \end{cases}$$

where $q$ is any element in $P$.

Define a mapping $\phi$ from $P$ into $S$ s.t.

$$\phi(p) = ((0,0),\ (e,p))\rho \ \ \forall p \in P$$

Therefore,

$$((n,k),\ (g,p))\rho = \begin{cases} ((n,k),(g,p))\phi(q) & \text{if } k>0 \\ ((n,0),(g,p))\phi(p) & \text{if } k=0 \end{cases}$$

where $q$ is any element in $P$. Then let $\rho = \rho_\phi$.

($\Leftarrow$) Let $((n,k),(g,p))$ and $((r,s),(h,q)) \in S$. There are four cases.

Case (i): If $r \geq k$ and $s > 0$

$$\begin{aligned} ((n,k),(g,p))[((r,s),(h,q))\rho_\phi] &= ((n,k),(g,p))[((r,s),(h,q))\phi(z)] \ \ z \in P. \\ &= [((n,k),(g,p))((r,s),(h,q))]\phi(z) \\ &= ((n+r-k,s),(g\theta^{r-k}h,q))\phi(z) \\ &= ((n+r-k,s),(g\theta^{r-k}h,q))\rho_\phi \ \ \text{since } s>0 \\ &= [((n,k),(g,p))((r,s),(h,q))]\rho_\phi. \end{aligned}$$

Case (ii): If $r < k$ and $s > 0$

$$
\begin{aligned}
&((n,k),(g,p))[((r,s),(h,q))\rho_\phi] \\
&\qquad = ((n,k),(g,p))[((r,s),(h,q))\phi(z)], \quad z \in P \\
&\qquad = [((n,k),(g,p))((r,s),(h,q))]\phi(z) \\
&\qquad = ((n,k+s-r),(g(h\theta^{k-r}),p(h\theta^{k-r-1}\gamma)))\phi(z) \\
&\qquad = ((n,k+s-r),(g(h\theta^{k-r}),p(h\theta^{k-r-1}\gamma)))\rho_\phi \quad \text{since } k+s-r>0 \\
&\qquad = [((n,k),(g,p))((r,s),(h,q))]\rho_\phi.
\end{aligned}
$$

<u>Case (iii)</u>: If $r \geq k$ and $s = 0$

$$
\begin{aligned}
((n,k),(g,p))[((r,0),(h,q))\rho_\phi] &= ((n,k),(g,p))[((r,0),(h,q))\phi(q)] \\
&= [((n,k),(g,p))((r,0),(h,q))]\phi(q) \\
&= ((n+r-k,0),(g\theta^{r-k}h,q))\phi(q) \\
&= ((n+r-k,0),(g\theta^{r-k}h,q))\rho_\phi \\
&= [((n,k),(g,p))((r,0),(h,q))]\rho_\phi.
\end{aligned}
$$

<u>Case (iv)</u> If $r < k$ and $s = 0$

$$
\begin{aligned}
((n,k),(g,p))[((r,0),(h,q))\rho_\phi] &= ((n,k),(g,p))[((r,0),(h,q))\phi(q)] \\
&= [((n,k),(g,p))((r,0),(h,q))]\phi(q) \\
&= ((n,k-r),(g(h\theta^{k-r}),p(h\theta^{k-r-1}\gamma)))\phi(q) \\
&= ((n,k-r),(g(h\theta^{k-r}),p(h\theta^{k-r-1}\gamma)))\rho_\phi \quad \text{since } k-r>0 \\
&= [((n,k),(g,p))((r,0),(h,q))]\rho_\phi.
\end{aligned}
$$

**Lemma 2.2** *Every left translation of an $E$-bisimple semigroup $S$ is an inner left translation of $S$.*

Proof: Let $\lambda$ be a left translation of an $E$-bisimple semigroup $S$. Since for any $p \in P$, we have

$$((n,k),(g,q)) = ((0,0),(e,p))((n,k),(g,q)) \quad \forall((n,k),(g,q)) \in S.$$

So, fix $p \in P$, and let $t = \lambda((0,0),(e,p))$. Therefore,

$$\begin{aligned}
\lambda((n,k),(g,q)) &= \lambda[((0,0),(e,p))((n,k),(g,q))] \\
&= [\lambda((0,0),(e,p))]((n,k),(g,q)) \\
&= t((n,k),(g,q)) \\
&= \lambda_t((n,k),(g,q)) \qquad \forall((n,k),(g,q)) \in S.
\end{aligned}$$

Therefore

$$\lambda = \lambda_t \text{ is an inner left translation of } S.$$

**Lemma 2.3** *Let $S$ be an $E$-bisimple semigroup. A right translation $\rho_\phi$ of $S$ and a left translation $\lambda_t$ of $S$ are linked if and only if*

$$\phi(q)((0,0),(e,p)) = t((0,0),(e,p)) \quad \forall p,q \in P.$$

Proof: Let $\rho_\phi$ be a right translation of an $E$-bisimple semigroup $S$ and let $\lambda_t$ be a left translation of $S$.

($\Leftarrow$) Suppose that $\phi(q)((0,0),(e,p)) = t((0,0),(e,p))$ $\forall p,q \in P$.

If $k > 0$, $z \in P$

$$\begin{aligned}
[((n,k),(g,p))\rho_\phi]((r,s),(h,q)) &= [((n,k),(g,p))\phi(z)]((r,s),(h,q)) \\
&= [((n,k),(g,p))\phi(z)][((0,0)(e,u))((r,s),(h,q))]
\end{aligned}$$

$$
\begin{aligned}
&= ((n,k),(g,p))[\phi(z)((0,0),(e,u))((r,s),(h,q))] \\
&= ((n,k),(g,p))[t((0,0),(e,u))(r,s),(h,q))] \\
&= ((n,k),(g,p))[t((r,s),(h,q))] \\
&= ((n,k),(g,p))[\lambda_t((r,s),(h,q))].
\end{aligned}
$$

If $k=0$

$$
\begin{aligned}
[((n,0),(g,p))\rho_\phi]((r,s),(h,q)) &= [((n,0),(g,p))\phi(p)]((r,s),(h,q)) \\
&= [((n,0),(g,p))\phi(p)][((0,0),(e,u))((r,s),(h,q))] \\
&= ((n,0),(g,p))[\phi(p)((0,0),(e,u))((r,s),(h,q))] \\
&= ((n,0),(g,p))[t((0,0),(e,u))((r,s),(h,q))] \\
&= ((n,0),(g,p))[t((r,s),(h,q))] \\
&= ((n,0),(g,p))[\lambda_t((r,s),(h,q))]
\end{aligned}
$$

Hence $\rho_\phi$ and $\lambda_t$ are linked.

($\Rightarrow$) Suppose that $\rho_\phi$ and $\lambda_t$ are linked; therefore,

$$\forall x,y \in S,\ \ x(\lambda_t y) = (x\rho_\phi)y.$$

Let $x = ((0,0),(e,q))$ and $y = ((0,0),(e,p))$. So,

$$((0,0),(e,q))[\lambda_t((0,0),(e,p))] = [((0,0),(e,q))\rho_\phi]((0,0),(e,p))$$

$$\Rightarrow ((0,0),(e,q))[t((0,0),(e,p))] = \phi(q)((0,0),(e,p))$$

i.e.,

$$t((0,0),(e,p)) = \phi(q)((0,0),(e,p))$$

and since $p$ and $q$ are arbitrary elements of $P$, the result is true $\forall\ p,q \in P$.

In the following two lemmas, we want to get a better form of the right translation of $S$ linked to some left translation $\lambda_t$ of $S$, by studying the mapping $\phi$ and giving some limits of the image of it.

**Lemma 2.4** *Let $\rho_\phi$ be a right translation of an E-bisimple semigroup $S$, then* $((0,1),(e,p))\phi(q) = ((0,1),(e,p))\phi(u) \quad \forall\, p \in K$ *and* $\forall\, q, u \in P$.

Proof: Consider the given,

$$\begin{aligned}
\Rightarrow ((0,1),(e,p))\phi(q) &= ((0,1),(e,p))[((0,0),(e,q))\rho_\phi] \\
&= [((0,1),(e,p))((0,0),(e,q))]\rho_\phi \\
&= ((0,1),(e,p))\rho_\phi \\
&= [((0,1),(e,p))((0,0),(e,u))]\rho_\phi \\
&= ((0,1)(e,p))[((0,0),(e,u))\rho_\phi] \\
&= ((0,1),(e,p))\phi(u)
\end{aligned}$$

and since $p, q$ and $u$ are arbitrary elements, the result is true $\forall p \in K$ and $q, u \in P$.

**Lemma 2.5** *Let $S$ be an E-bisimple semigroup and let $\rho_\phi$ be a right translation of $S$ linked to some left translation $\lambda_t$ of $S$. Then either $\phi$ is a constant mapping i.e., $\phi(p) = v\ \forall p \in P$ where $v = ((a_\phi, b_\phi), (g_\phi, y_\phi))$ and hence $\rho_\phi = \rho_v$ is an inner right translation or $\phi(p) = ((0,0), (g_\phi, p\delta_\phi))$ where $\delta_\phi : P \to P$ is a non-constant mapping and in this case we may write $\rho_\phi$ as $\rho_{(g_\phi,\delta_\phi)}$ where $(g_\phi, \delta_\phi)(p) =$ $((0,0), (g_\phi, p\delta_\phi))$.*

<u>Proof:</u> Since $\rho_\phi$ is linked to $\lambda_t$, by Lemma 2.3, we have

$$\phi(q)((0,0),(e,p)) = t((0,0),(e,p)) = \phi(u)((0,0),(e,p)) \;\; \forall p,q,u \in P.$$

Let

$$\phi(q) = ((a_1,b_1),(x_1,y_1)) \text{ and } \phi(u) = ((a_2,b_2),(x_2,y_2))$$

$$\Rightarrow ((a_1,b_1),(x_1,y_1))((0,0),(e,p)) = ((a_2,b_2),(x_2,y_2))((0,0),(e,p)).$$

Thus, by Warne's structure theorem, we get

$$b_1 > 0 \Rightarrow b_2 > 0 \text{ and } ((a_1,b_1),(x_1,y_1)) = ((a_2,b_2),(x_2,y_2)).$$

Hence, if $\exists q \in P$, s.t.

$$\phi(q) = ((a_1,b_1),(x_1,y_1)) \text{ with } b_1 > 0$$

$$\Rightarrow \phi(p) = ((a_1,b_1),(x_1,y_1)) \quad \forall p \in P$$

Therefore

$\phi$ is a constant mapping.

Thus $\rho_\phi = \rho_{v=((a_1,b_1),(x_1,y_1))}$ is an inner right translation.

But $b_1 = 0 \;\Rightarrow b_2 = 0$ and

$$((a_1,0),(x_1,y_1))((0,0),(e,p)) = ((a_1,0),(x_1,p))$$

&

$$((a_2,0),(x_2,y_2))((0,0),(e,p)) = ((a_2,0),(x_2,p))$$

i.e.,

$$a_1 = a_2,\; b_1 = b_2 = 0,\; x_1 = x_2.$$

Therefore

$$\phi(p) = ((a_1,0),(x_1,y)) \ \forall\, p \in P, \text{ where } y \in P$$

which we may write as $\phi(p) = ((a_1,0),\ (x_1, p\delta_\phi))$ where $\delta_\phi : P \to P$ is a mapping.

Now suppose that $a_1 > 0$, so, by Lemma 2.4 we get

$$((0,1),(e,p))((a_1,0),(x_1,q\delta_\phi)) = ((0,1),(e,p))((a_1,0),(x_1,u\delta_\phi)) \ \forall p \in K \text{ and } \ \forall q,u \in P.$$

$$\Rightarrow ((a_1-1,0),(x_1,q\delta_\phi)) = ((a_1-1,0),(x_1,u\delta_\phi)) \ \forall q,u \in P$$

i.e.,

$$q\delta_\phi = u\delta_\phi \quad \forall q,u \in P$$

Therefore

$$\delta_\phi \text{ is a constant mapping}$$

$$\Rightarrow \phi \ \text{ is a constant mapping}$$

Hence $\rho_\phi = \rho_{v=((a_1,0),(x_1,y_1))}$ is an inner right translation.

Finally, if $\phi(p) = ((0,0),(x_1,p\delta_\phi)) \ \forall p \in P$, where $\delta_\phi : P \to P$ is a mapping, then either $\delta_\phi$ is a constant mapping i.e., $p\delta_\phi = k \ \forall p \in P$, and in this case, $\rho_\phi = \rho_{v=((0,0),(x_1,k))}$ is an inner right translation, or $\delta_\phi$ is a nonconstant mapping which we may write $\rho_\phi$ in this case as $\rho_{(x_1,\delta_\phi)}$, where $(x_1,\delta_\phi)(p) = ((0,0),(x_1,p\delta_\phi)) \ \forall p \in P$.

**Lemma 2.6** *Let $S$ be an E-bisimple semigroup and let $\lambda_t$ and $\lambda_s$ be two left translations of $S$. Then,*

$$\lambda_s = \lambda_t \Leftrightarrow s((0,0),(e,p)) = t((0,0),(e,p)) \ \forall p \in P.$$

Proof: Suppose that $\lambda_s = \lambda_t$. Therefore

$$\lambda_s((n,k),(g,p)) = \lambda_t((n,k),(g,p)) \quad \forall((n,k),(g,p)) \in S.$$

In particular,

$$\lambda_s((0,0),(e,p)) = \lambda_t((0,0),(e,p)) \quad \forall p \in P.$$

$$\Rightarrow s((0,0),(e,p)) = t((0,0),(e,p)) \quad \forall p \in P.$$

Multiplying both sides by $((n,k),(g,q))$ from the right, we get

$$s((0,0),(e,p))((n,k),(g,q)) = t((0,0),(e,p))((n,k),(g,q)) \quad \forall p \in P$$

$$\Rightarrow s((n,k),(g,q)) = t((n,k),(g,q))$$

$$\Rightarrow \lambda_s((n,k),(g,q)) = \lambda_t((n,k),(g,q))$$

and since $((n,k),(g,q))$ is an arbitrary element in $S$, we conclude that

$$\lambda_s = \lambda_t.$$

**Theorem 2.7** *Let $S = (G,P,K,\theta,\gamma)$ be an E-bisimple semigroup, and let $W = \{(g,\delta) | g \in G$ and $\delta : P \to P$ is a nonconstant mapping$\}$. Let $\bar{S}$ be the translational hull of $S$. Then, $\bar{S} = S \cup W$ under the multiplication*

$$(g_1,\delta_1)\cdot(g_2,\delta_2) = \begin{cases} (g_1g_2, \delta_1 o \delta_2) & \text{if } \delta_1 o \delta_2 \text{ is not constant} \\ ((0,0),(g_1g_2,k)) & \text{if } (p)\delta_1 o \delta_2 = k \ \forall p \in P \end{cases}$$

$$((a,b),(h,q))\cdot(g,\delta) = \begin{cases} ((a,b),(h(g\theta^b), q(g\theta^{b-1}\gamma))) & \text{if } b > 0 \\ ((a,0),(hg, q\delta)) & \text{if } b = 0 \end{cases}$$

$$(g,\delta)\cdot((a,b),(h,q)) = ((a,b),((g\theta^a)h,q))$$

*where juxtaposition denotes the multiplication of $G$ and $o$ denotes iteration of mappings,*

*and $v_1 \cdot v_2 = v_1 v_2$ where $v_1, v_2 \in S$ and juxtaposition denotes the multiplication of $S$.*

Proof: Since, by Lemma 2.5, any right translation of an $E$-bisimple semigroup $S$ which is linked to some left translation is either an inner right translation $\rho_v$ where $v \in S$, or of the form $\rho_{(g,\delta)}$ where $(g,\delta)(p) = ((0,0),(g,p\delta))$ $\forall p \in P$, s.t. $\delta : P \to P$ is a nonconstant mapping.

So, there are four cases to determine the multiplication of any two right translations; each one is linked to some left translation.

Firstly, suppose that both $\rho_{(g_1,\delta_1)}$ and $\rho_{(g_2,\delta_2)}$ are not inner right translations, then since

$$\begin{aligned}
\left(((0,0),(e,p))\rho_{(g_1,\delta_1)}\right)\rho_{(g_2,\delta_2)} &= ((0,0),(g_1,p\delta_1))\rho_{(g_2,\delta_2)} \\
&= ((0,0),(g_1,p\delta_1))((0,0),(g_2,(p\delta_1)\delta_2)) \\
&= \begin{cases} ((0,0),(g_1g_2,(p)\delta_1\delta_2)) & \text{if } \delta_1\delta_2 \text{ is not constant} \\ ((0,0),(g_1g_2,k_0)) & \text{if } (p)\delta_1\delta_2 = k_0 \ \forall p \in P. \end{cases}
\end{aligned}$$

Thus, $\forall((n,k),(g,q)) \in S$, we have

$$\left(((n,k),(g,q))\rho_{(g_1,\delta_1)}\right)\rho_{(g_2,\delta_2)}$$

$$= \begin{cases} \left([((n,k),(g,q))((0,0),(e,p))]\,\rho_{(g_1,\delta_1)}\right)\rho_{(g_2,\delta_2)} & \text{if } k > 0 \\ \left([((n,0),(g,q))((0,0),(e,q))]\,\rho_{(g_1,\delta_1)}\right)\rho_{(g_2,\delta_2)} & \text{if } k = 0 \end{cases}$$

$$= \begin{cases} ((n,k),(g,q))\left(\left(((0,0),(e,p))\rho_{(g_1,\delta_1)}\right)\rho_{(g_2,\delta_2)}\right) & \text{if } k>0 \\ ((n,0),(g,q))\left(\left(((0,0)(e,q))\rho_{(g_1,\delta_1)}\right)\rho_{(g_2,\delta_2)}\right) & \text{if } k=0 \end{cases}$$

$$= \begin{cases} \begin{cases} ((n,k),(g,q))((0,0),(g_1g_2,(p)\delta_1\delta_2)) & \text{if } k>0 \text{ and } \delta_1\delta_2 \text{ is not constant} \\ ((n,k),(g,q))((0,0),(g_1g_2,k_0)) & \text{if } k>0 \text{ and } (p)\delta_1\delta_2 = k_0,\ \forall p \in P \end{cases} \\ \begin{cases} ((n,0),(g,q))((0,0),(g_1g_2,(q)\delta_1\delta_2)) & \text{if } k=0 \text{ and } \delta_1\delta_2 \text{ is not constant} \\ ((n,0),(g,q))((0,0),(g_1g_2,k_0)) & \text{if } k=0 \text{ and } (q)\delta_1\delta_2 = k_0,\ \forall q \in P. \end{cases} \end{cases}$$

$$= \begin{cases} \begin{cases} ((n,k),(g,q))((0,0),(g_1g_2,(p)\delta_1\delta_2)) & \text{if } k>0 \\ ((n,0),(g,q))((0,0),(g_1g_2,(q)\delta_1\delta_2)) & \text{if } k=0 \end{cases} & \text{if } \delta_1\delta_2 \text{ is not constant} \\ \begin{cases} ((n,k),(g,q))((0,0),(g_1g_2,k_0)) & \text{if } k>0 \\ ((n,0),(g,q))((0,0),(g_1g_2,k_0)) & \text{if } k=0 \end{cases} & \text{if } (p)\delta_1\delta_2 = k_0\ \forall p \in P. \end{cases}$$

$$= \begin{cases} \begin{cases} ((n,k),(g,q))(g_1g_2,\delta_1\delta_2)(p) & \text{if } k>0 \\ ((n,0),(g,q))(g_1g_2,\delta_1\delta_2)(q) & \text{if } k=0 \end{cases} & \text{if } \delta_1\delta_2 \text{ is not constant} \\ ((n,k),(g,q))\rho_{((0,0),(g_1g_2,k_0))} & \text{if } (p)\delta_1\delta_2 = k_0\ \forall p \in P \end{cases}$$

$$= ((n,k),(g,q)) \begin{cases} \rho_{(g_1g_2,\delta_1\delta_2)} & \text{if } \delta_1\delta_2 \text{ is not constant} \\ \rho_{((0,0),(g_1g_2,k_0))} & \text{if } (p)\delta_1\delta_2 = k_0,\ \forall p \in P. \end{cases}$$

whence

$$\rho_{(g_1,\delta_1)} \cdot \rho_{(g_2,\delta_2)} = \begin{cases} \rho_{(g_1g_2,\delta_1\delta_2)} & \text{if } \delta_1\delta_2 \text{ is not constant} \\ \rho_{((0,0),(g_1g_2,k))} & \text{if } (p)\delta_1\delta_2 = k\ \forall p \in P. \end{cases}$$

Secondly, suppose that $\rho_{v_1=((e_1,b_1),(h_1,g_1))}$ is an inner right translation and $\rho_{(g_2,\delta_2)}$ is not.

Since

$$[((0,0)(e,p))\rho_{v_1}]\,\rho_{(g_2,\delta_2)} = ((a_1,b_1),(h_1,q_1))\rho_{(g_2,\delta_2)}$$

$$= \begin{cases} ((a_1,b_1),(h_1,q_1))((0,0)(g_2,p\delta_2)) & \text{if } b_1 > 0 \\ ((a_1,0),(h_1,q_1))((0,0),(g_2,q_1\delta_2)) & \text{if } b_1 = 0 \end{cases}$$

$$= \begin{cases} ((a_1,b_1),(h_1(g_2\theta^{b_1}),q_1(g_2\theta^{b_1-1}\gamma))) & \text{if } b_1 > 0 \\ ((a_1,0),(h_1g_2,q_1\delta_2)) & \text{if } b_1 = 0 \end{cases}$$

Hence, we can show easily that

$$\rho_{v_1=((a_1,b_1),(h_1,q_1))} \cdot \rho_{(g_2,\delta_2)} = \begin{cases} \rho_{((a_1,b_1),(h_1(g_2\theta^{b_1}),q_1(g_2\theta^{b_1-1}\gamma)))} & \text{if } b_1 > 0 \\ \rho_{((a_1,0),(h_1g_2,q_1\delta_2))} & \text{if } b_1 = 0 \end{cases}$$

Thirdly, suppose that $\rho_{(g_1,\delta_1)}$ is not an inner right translation and $\rho_{v_2=((a_2,b_2),(h_2,q_2))}$ is an inner right transalation, then since

$$\begin{aligned} \left(((0,0),(e,p))\rho_{(g_1,\delta_1)}\right)\rho_{v_2=((a_2,b_2),(h_2,q_2))} & \\ &= ((0,0),(g_1,p\delta_1))\rho_{((a_2,b_2),(h_2,q_2))} \\ &= ((0,0),(g_1,p\delta_1))((a_2,b_2),(h_2,q_2)) \\ &= ((a_2,b_2),((g_1\theta^{a_2})h_2,q_2)) \end{aligned}$$

Thus, it can be shown easily that

$$\rho_{(g_1,\delta_1)} \cdot \rho_{v_2=((a_2,b_2),(h_2,q_2))} = \rho_{((a_2,b_2),((g_1\theta^{a_2})h_2,q_2))}.$$

Finally, if both $\rho_{v_1}$ and $\rho_{v_2}$ are inner right translations, it is very clear that $\rho_{v_1} \cdot \rho_{v_2} = \rho_{v_1v_2}$ where $v_1v_2$ is multiplied as in $S$.

Next, note that if $(\rho_v, \lambda_s) \in \bar{S}$, then by Lemma 2.3,

$$v((0,0),(e,p)) = s((0,0),(e,p)) \qquad \forall p \in P.$$

$$\Rightarrow \text{ by Lemma 2.6 }, \qquad \lambda_v = \lambda_s.$$

Thus,

$$(\rho_v, \lambda_s) = (\rho_v, \lambda_v).$$

So, every element in $\bar{S}$ of the form $(\rho_v, \lambda_s)$, we may write it as $(\rho_v, \lambda_v)$.

Let $W = \{(g,\delta) | g \in G$ and $\delta : P \to P$ is a nonconstant mapping$\}$.

Now, we want to set an isomorphism between $\bar{S}$ and $S \cup W$, so, define a mapping $T : \bar{S} \to S \cup W$ s.t.

$$T(\rho_v, \lambda_v) = v \quad \text{and} \quad T(\rho_{(g,\delta)}, \lambda_s) = (g, \delta).$$

$T$ is well defined since if

$$\begin{aligned} (\rho_{v_1}, \lambda_{v_1}) &= (\rho_{v_2}, \lambda_{v_2}) \\ &\Rightarrow \rho_{v_1} = \rho_{v_2} \\ &\Rightarrow v_1 = v_2 \Rightarrow T(\rho_{v_1}, \lambda_{v_1}) = T(\rho_{v_2}, \lambda_{v_2}) \end{aligned}$$

and if $(\rho_{(g_1,\delta_1)}, \lambda_{s_1}) = (\rho_{(g_2,\delta_2)}, \lambda_{s_2})$

$$\begin{aligned} &\Rightarrow \rho_{(g_1,\delta_1)} = \rho_{(g_2,\delta_2)} \\ &\Rightarrow (g_1,\delta_1)(p) = (g_2,\delta_2)(p) \ \forall p \in P \\ &\Rightarrow (g_1,\delta_1) = (g_2,\delta_2) \\ &\Rightarrow T(\rho_{(g_1,\delta_1)}, \lambda_{s_1}) = T(\rho_{(g_2,\delta_2)}, \lambda_{s_2}). \end{aligned}$$

Note that $(\rho_{(g,\delta)}, \lambda_s) \neq (\rho_v, \lambda_v)$. Suppose not, i.e.,

$$\begin{aligned}(\rho_{(g,\delta)}, \lambda_s) &= (\rho_v, \lambda_v) \\ &\Rightarrow \rho_{(g,\delta)} = \rho_v \quad \text{contradiction}\end{aligned}$$

since $\rho_v$ is an inner right translation but $\rho_{(g,\delta)}$ is not because $\delta$ is not a constant mapping.

$T$ is a homomorphism because

$$\begin{aligned}
&T((\rho_{(g_1,\delta_1)}, \lambda_{s_1}) \cdot (\rho_{(g_2,\delta_2)}, \lambda_{s_2})) \\
&\quad = T(\rho_{(g_1,\delta_1)} \cdot \rho_{(g_2,\delta_2)}, \lambda_{s_1}\lambda_{s_2}) \\
&\quad = \begin{cases} T(\rho_{(g_1g_2,\delta_1\delta_2)}, \lambda_{s_1s_2}) & \text{if } \delta_1\delta_2 \text{ is not constant} \\ T(\rho_{((0,0),(g_1g_2,k))}, \lambda_{s_1s_2}) & \text{if } (p)\delta_1\delta_2 = k \ \forall p \in P \end{cases} \\
&\quad = \begin{cases} (g_1g_2, \delta_1\delta_2) & \text{if } \delta_1\delta_2 \text{ is not constant} \\ ((0,0),(g_1g_2,k)) & \text{if } (p)\delta_1\delta_2 = k \ \forall p \in P. \end{cases} \\
&\quad = (g_1, \delta_1) \cdot (g_2, \delta_2) \\
&\quad = T(\rho_{(g_1,\delta_1)}, \lambda_{s_1}) \cdot T(\rho_{(g_2,\delta_2)}, \lambda_{s_2}).
\end{aligned}$$

and since

$$\begin{aligned}
&T((\rho_{((a,b),(h,q))}, \lambda_{((a,b),(h,q))}) \cdot (\rho_{(g,\delta)}, \lambda_s)) \\
&\quad = T(\rho_{((a,b),(h,q))} \cdot \rho_{(g,\delta)}, \lambda_{((a,b),(h,q))}\lambda_s) \\
&\quad = \begin{cases} T(\rho_{((a,b),(h(g\theta^b),q(g\theta^{b-1}\gamma)))}, \lambda_{((a,b),(h,q))s}) & \text{if } b > 0 \\ T(\rho_{((a,0),(hg,q\delta))}, \lambda_{((a,0),(h,q))s}) & \text{if } b = 0 \end{cases} \\
&\quad = \begin{cases} ((a,b),(h(g\theta^b), q(g\theta^{b-1}\gamma))) & \text{if } b > 0 \\ ((a,0),(hg,q\delta)) & \text{if } b = 0 \end{cases}
\end{aligned}$$

$$= ((a,b),(h,q)) \cdot (g,\delta)$$

$$= T(\rho_{((a,b),(h,q))}, \lambda_{((a,b),(h,q))}) \cdot T(\rho_{(g,\delta)}, \lambda_s).$$

Also, since

$$\begin{aligned}
T((\rho_{(g,\delta)}, \lambda_s) \cdot (\rho_{((a,b),(h,q))}, \lambda_{((a,b),(h,q))})) \\
&= T(\rho_{(g,\delta)} \cdot \rho_{((a,b),(h,q))}, \lambda_s \lambda_{((a,b),(h,q))}) \\
&= T(\rho_{((a,b),((g\theta^a)h,q))}, \lambda_{s((a,b),(h,q))}) \\
&= ((a,b),((g\theta^a)h,q)) \\
&= (g,\delta) \cdot ((a,b),(h,q)) \\
&= T(\rho_{(g,\delta)}, \lambda_s) \cdot T(\rho_{((a,b),(h,q))}, \lambda_{((a,b),(h,q))}).
\end{aligned}$$

Finally because

$$\begin{aligned}
T((\rho_v, \lambda_v) \cdot (\rho_t, \lambda_t)) &= T(\rho_{vt}, \lambda_{vt}) \\
&= vt = v \cdot t \\
&= T(\rho_v, \lambda_v) \cdot T(\rho_t, \lambda_t)
\end{aligned}$$

So, we have proved that $T$ is a homomorphism.

Now, we want to prove that $T$ is $1-1$. Suppose that

$$\begin{aligned}
T(\rho_v, \lambda_v) &= T(\rho_t, \lambda_t) \\
&\Rightarrow v = t \\
&\Rightarrow (\rho_v, \lambda_v) = (\rho_t, \lambda_t).
\end{aligned}$$

If

$$T(\rho_{(g_1,\delta_1)}, \lambda_{s_1}) = T(\rho_{(g_2,\delta_2)}, \lambda_{s_2})$$

$$\Rightarrow \quad (g_1, \delta_1) = (g_2, \delta_2)$$

$$\Rightarrow \quad \rho_{(g_1,\delta_1)} = \rho_{(g_2,\delta_2)} \quad \text{and ,}$$

$$(g_1, \delta_1)(q)((0,0),(e,p)) = (g_2, \delta_2)(q)((0,0),(e,p)) \;\; \forall p, q \in P$$

So, by Lemma 2.3, we get

$$s_1((0,0),(e,p)) = s_2((0,0),(e,p)) \;\; \forall p \in P$$

Therefore

$$\lambda_{s_1} = \lambda_{s_2} \quad \text{(by Lemma 2.6)}$$

$$\Rightarrow (\rho_{(g_1,\delta_1)}, \lambda_{s_1}) = (\rho_{(g_2,\delta_2)}, \lambda_{s_2}).$$

Note that $T(\rho_{(g,\delta)}, \lambda_s) \neq T(\rho_v, \lambda_v) \;\forall T(\rho_{(g,\delta)}, \lambda_s)$ and $T(\rho_v, \lambda_v) \in T(\bar{S})$, since if

$$\exists T(\rho_{(g,\delta)}, \lambda_s) \text{ and } T(\rho_v, \lambda_v) \in T(\bar{S})$$

$$\text{s.t. } T(\rho_{(g,\delta)}, \lambda_s) = T(\rho_v, \lambda_v)$$

$$\Rightarrow (g, \delta)(p) = v \qquad \forall\, p \in P$$

$$\Rightarrow \delta \text{ is constant} \quad \text{contradiction .}$$

Therefore $T$ is $1-1$.

Now, we want to show that $T$ is onto.

Since $\forall v \in S$, $\rho_v$ and $\lambda_v$ are linked. Therefore

$$(\rho_v, \lambda_v) \in \bar{S} \text{ s.t. } T(\rho_v, \lambda_v) = v$$

and since $\forall (g, \delta) \in W$

$$\exists\, \rho_{(g,\delta)} \text{ and } \lambda_s \text{ where } s = ((0,0),(g,k_0)).$$

$\rho_{(g,\delta)}$ and $\lambda_s$ are linked since

$$\begin{aligned}(g,\delta)(q)((0,0)(e,p)) &= ((0,0),(g,q\delta))((0,0),(e,p)) \\ &= ((0,0),(g,p)) \\ &= ((0,0),(g,k_0))((0,0),(e,p)) \\ &= s((0,0),(e,p)) \ \forall q,p \in P.\end{aligned}$$

Therefore,

$$(\rho_{(g,\delta)}, \lambda_s) \in \bar{S} \text{ s.t. } T(\rho_{(g,\delta)}, \lambda_s) = (g,\delta)$$

Therefore, $T$ is an isomorphism. Hence,

$$\bar{S} = S \cup W$$

# CHAPTER 3

# IDEAL EXTENSION OF $E$-BISIMPLE SEMIGROUP THEOREMS

Clifford gave general means for finding all possible extensions of a weakly reductive semigroup $S$ by a semigroup $T$ with zero. However, as in group theory, these means are of a theoretical nature, and to carry them out to give an explicit determination of the extensions for particular classes of semigroups is usually difficult.

The two main results of this chapter are the determination of all extensions of an $E$-bisimple semigroup by a completely 0-simple semigroup and the determination of all extensions of a Brandt semigroup by an $E$-bisimple semigroup.

**Theorem 3.1** *Let $S = (G, K, P, \theta, \gamma)$ be an $E$-bisimple semigroup and let $T = M^0(H, F, \Lambda, Q)$ be a completely 0-simple semigroup. Let the following mappings be given*

$$\psi : F \to I^0, \xi : \Lambda \to I^0, \tau : \Lambda \to P \cup K, \ \alpha : F \to G \text{ and } \beta : \Lambda \to G$$

*and $\eta$ be a homomorphism of $H$ into $G$ such that $q_{mi}\eta \neq 0$ implies $m\xi = i\psi$ and*

*$(m\beta)(i\alpha) = q_{mi}\eta$. Then $\varphi$ defined on $T^*$ (i.e., $T\backslash 0$) by*

$$(a,i,m)\varphi = ((i\psi, m\xi), (i\alpha a\eta m\beta, m\tau))$$

*is a partial homomorphism of $T^*$ into $S$. Conversely, every partial homomorphism of $T^*$ into $S$ is obtained in this fashion.*

Proof: For the direct part, consider the given, let $(a,i,m),(b,j,n) \in T^*$ and suppose that $q_{mj} \neq 0$ which implies that $m\xi = j\psi$ and $(m\beta)(j\alpha) = q_{mj}\eta$.

Now we want to show that $\varphi$ is a partial homomorphism of $T^*$ into $S$. Since

$$\begin{aligned}
((a,i,m)(b,j,n))\varphi &= (aq_{mj}b,i,n)\varphi \\
&= ((i\psi,n\xi),(i\alpha(aq_{mj}b)\eta n\beta,n\tau)) \\
&= ((i\psi,n\xi),(i\alpha a\eta q_{mj}\eta b\eta n\beta,n\tau)) \\
&= ((i\psi,n\xi),(i\alpha a\eta m\beta j\alpha b\eta n\beta,n\tau)) \\
&= ((i\psi,m\xi),(i\alpha a\eta m\beta,m\tau))((j\psi,n\xi),(j\alpha b\eta n\beta,n\tau)) \\
&= (a,i,m)\varphi(b,j,n)\varphi.
\end{aligned}$$

Hence $\varphi$ is a partial homomorphism of $T^*$ into $S$.

Conversely, suppose that $\varphi$ is a partial homomorphism of $T^*$ into $S$. Since $\varphi$ maps $\mathcal{R}$-classes of $T^*$ into $\mathcal{R}$-classes of $S$ and maps $\mathcal{L}$-classes of $T^*$ into $\mathcal{L}$-classes of $S$ by lemma 1.3, and since the set of $\mathcal{R}$-classes in $T^*$ is $\{R_i | i \in F\}$ where

$$R_i = \{(a,i,m) | a \in H \quad \text{and} \quad m \in \Lambda\},$$

the set of $\mathcal{R}$-classes in $S$ is $\{R_n | n \in I^0\}$ where

$$R_n = \{((n,k),(g,q)) | g \in G \text{ and either } k = 0, q \in P \text{ or } k \in N,\ q \in K\},$$

the set of $\mathcal{L}$-classes in $T^*$ is $\{L_m | m \in \Lambda\}$ where

$$L_m = \{(a,i,m) | a \in H \quad \text{and} \quad i \in F\}$$

and the set of $\mathcal{L}$-classes in $S$ is $\{L_{k,q} |$ either $k = 0,\ q \in P$ or $k \in N,\ q \in K\}$ where

$$L_{k,q} = \left\{((n,k),(g,q)) | n \in I^0,\ g \in G\right\}.$$

Then we may write

$$(a,i,m)\varphi = ((i\psi, m\xi), (g, m\tau))$$

where $g \in G$ and $\psi : F \to I^0$, $\xi : \Lambda \to I^0$ and $\tau : \Lambda \to P \cup K$ are mappings.

Now, by the proof of the Rees's theorem of completely 0-simple semigroups, we may assume that $1 \in F \cap \Lambda$ and $q_{11} = e_H$, the identity of $H$.

Define the mappings $\alpha : F \to G$ and $\beta : \Lambda \to G$ such that

$$(e,i,1)\varphi = ((i\psi, 1\xi), (i\alpha, 1\tau))$$

and

$$(e,1,m)\varphi = ((1\psi, m\xi), (m\beta, m\tau)).$$

So, $(e,1,1)$ is an idempotent element of $T$, and since the general form of idempotent elements of $S$ is $((n,n),(e_G,q))$ where $n \in I^0$ and $q \in P$ if $n = 0$ or

$q \in K$ if $n > 0$ by Lemma 1.12, $(e, 1, 1)\varphi = ((1\psi, 1\xi), (e_G, 1\tau))$ where $1\psi = 1\xi$ because $\varphi$ maps idempotent elements of $T$ into idempotent elements of $S$ by Lemma 1.1.

Define the mapping $\eta : H \to G$ such that

$$(a, 1, 1)\varphi = ((1\psi, 1\xi), (a\eta, 1\tau)).$$

Now since

$$\begin{aligned}
(a, 1, 1)(b, 1, 1) &= (ab, 1, 1) \\
\Rightarrow (a, 1, 1)\varphi(b, 1, 1)\varphi &= (ab, 1, 1)\varphi \\
\Rightarrow ((1\psi, 1\xi), (a\eta, 1\tau))((1\psi, 1\xi), (b\eta, 1\tau)) &= ((1\psi, 1\xi), ((ab)\eta, 1\tau)) \\
\Rightarrow ((1\psi, 1\xi), ((a\eta)(b\eta), 1\tau)) &= ((1\psi, 1\xi), ((ab)\eta, 1\tau))
\end{aligned}$$

So, we get $(a\eta)(b\eta) = (ab)\eta$.

Therefore $\eta$ is a homomorphism of $H$ into $G$.

Also since

$$\begin{aligned}
(e, 1, m)(e, i, 1) &= (q_{mi}, 1, 1) \quad \text{if } q_{mi} \neq 0 \\
\Rightarrow (e, 1, m)\varphi(e, i, 1)\varphi &= (q_{mi}, 1, 1)\varphi \\
\Rightarrow ((1\psi, m\xi), (m\beta, m\tau))((i\psi, 1\xi), (i\alpha, 1\tau)) &= ((1\psi, 1\xi), (q_{mi}\eta, 1\tau))
\end{aligned}$$

So, by Warne's structure theorem of $E$-bisimple semigroups, it easily follows that $m\xi = i\psi$ and $(m\beta)(i\alpha) = q_{mi}\eta$ if $q_{mi} \neq 0$.

Finally, since

$$\begin{aligned}(a,i,m) &= (e,i,1)(a,1,1)(e,1,m)\\ \Rightarrow (a,i,m)\varphi &= (e,i,1)\varphi(a,1,1)\varphi(e,1,m)\varphi\\ &= ((i\psi,1\xi),(i\alpha,1\tau))((1\psi,1\xi),(a\eta,1\tau))((1\psi,m\xi),(m\beta,m\tau))\\ &= ((i\psi,1\xi),(i\alpha a\eta,1\tau))((1\psi,m\xi),(m\beta,m\tau))\\ &= ((i\psi,m\xi),(i\alpha a\eta m\beta,m\tau)).\end{aligned}$$

**Theorem 3.2** *Let $S=(G,P,K,\theta,\gamma)$ be an E-bisimple semigroup and let $T=M^0(H,F,\Lambda,Q)$ be a completely 0-simple semigroup. Let $i\to u_i$ and $m\to v_m$ be mappings of $F$ into $G$ and $\Lambda$ into $G$ respectively. Let $\omega$ be a homomorphism of $H$ into $G$ such that $q_{mj}w=v_mu_j$ if $q_{mj}\neq 0$, let $i\to\alpha_i$ and $m\to\beta_m$ be mappings of $F$ into $Y$ and $\Lambda$ into $Y$ respectively where $Y$ denotes the set of non-constant mappings of $P$, and let $a\to\gamma_a$ be a mapping of $H$ into $Y$ such that $\gamma_a\gamma_b=\gamma_{ab}$ for all $a,b\in H$, $\alpha_i\gamma_a\beta_m\in Y$ for all $i\in F$, $a\in H$ and $m\in\Lambda$, $\beta_m\alpha_j=\gamma_{q_{mj}}$ if $q_{mj}\neq 0$ and $\beta_m\alpha_j$ is a constant mapping of $P$ into $P$ if $q_{mj}=0$.*

*Let $V=S\cup T^*$ under the multiplication "o" defined as follows:*

$$(a,i,m)o(b,j,n)=\begin{cases}(a,i,m)(b,j,n) & \text{if } q_{mj}\neq 0\\ ((0,0),(u_ia\omega v_mu_jb\omega v_n,k_0)) & \text{if } q_{mj}=0\\ \quad\text{where } k_0=l_0\gamma_b\beta_n & \text{where } p\beta_m\alpha_j=l_0 \quad \forall p\in P\end{cases} \tag{5}$$

$$(a,i,m)o((n,k),(g,p))=((n,k),(((u_ia\omega v_m)\theta^n)g,p)) \tag{6}$$

$$((n,k),(g,p))o(a,i,m)=\begin{cases}((n,k),(g((u_ia\omega v_m)\theta^k),p((u_ia\omega v_m)\theta^{k-1}\gamma))) & \text{if } k>0\\ ((n,0),(g(u_ia\omega v_m),(p)\alpha_i\gamma_a\beta_m)) & \text{if } k=0\end{cases} \tag{7}$$

$$((n,k),(g,p))o((r,s),(h,q)) = ((n,k),(g,p))((r,s),(h,q)) \tag{8}$$

*where juxtaposition denotes the multiplications of $T$ and $S$. Then $(V,o)$ is an extension of $S$ by $T$.*

*Conversely, every extension of $S$ by $T$ is determined in the above manner or is given by a partial homomorphism and hence in this case an explicit multiplication is given by employing Theorem 3.1.*

Proof: We first establish the converse. Let $(V,o)$ be an extension of $S$ by $T$ and let $\bar{S}$ be the translational hull of $S$. Since, by lemma 1.13, $S$ is weakly reductive, there exists an extension $(\bar{V},*)$ of $\bar{S}$ by $T$ such that $(V,o)$ is a subsemigroup of $(\bar{V},*)$ by theorem 1.15. Let $e_G$ be the identity element of the group $G$ and $\delta_i$ be the identity mapping from $P$ into itself. Since, using Theorem 2.7, $(e_G,\delta_i)$ is the identity element of $\bar{S}$, $(\bar{V},*)$ is determined by a partial homomorphism $\pi$ from $T^*$ into $\bar{S}$ by virtue of theorem 1.14. Now since partial homomorphisms map $\mathcal{D}$-classes of $T^*$ into $\mathcal{D}$-classes of $\bar{S}$ by lemma 1.3 and since $T^*$ is a single $\mathcal{D}$-class and $S$ is a single $\mathcal{D}$-class in $\bar{S}$, then either $(T^*)\pi \subseteq S$ or $(T^*)\pi \subseteq \bar{S}\backslash S$. If $(T^*)\pi \subseteq S$, then $(V,o)$ is given by the partial homomorphism $\pi$ and the multiplication "$o$" of $V$ is thus determined by employing Theorem 3.1. But if $(T^*)\pi \subseteq \bar{S}\backslash S$ and since $\bar{S}\backslash S = G \times Y$ where $Y = \{\delta : P \to P | \delta \text{ is not constant}\}$ then we may write $(a,i,m)\pi = ((a,i,m)\varphi,(a,i,m)\psi)$ such that $\varphi : T^* \to G$ and $\psi : T^* \to Y$ are mappings. Let $(a,i,m)$, $(b,j,n) \in T^*$ and suppose that $q_{mj} \neq 0$. Now since $\pi$ is a partial homomorphism, then

$$\begin{aligned}
[(a,i,m)(b,j,n)]\pi &= (a,i,m)\pi(b,j,n)\pi \\
\Rightarrow (aq_{mj}b,i,n)\pi &= (a,i,m)\pi(b,j,n)\pi \\
\Rightarrow ((aq_{mj}b,i,n)\varphi,(aq_{mj}b,i,n)\psi) &= ((a,i,m)\varphi,(a,i,m)\psi)((b,j,n)\varphi,(b,j,n)\psi) \\
\Rightarrow ((aq_{mj}b,i,n)\varphi,(aq_{mj}b,i,n)\psi) &= ((a,i,m)\varphi(b,j,n)\varphi,(a,i,m)\psi(b,j,n)\psi) \\
\Rightarrow (aq_{mj}b,i,n)\varphi &= (a,i,m)\varphi(b,j,n)\varphi \\
\text{and } (aq_{mj}b,i,n)\psi &= (a,i,m)\psi(b,j,n)\psi \\
\Rightarrow [(a,i,m)(b,j,n)]\varphi &= (a,i,m)\varphi(b,j,n)\varphi \\
&\quad \text{i.e. } \varphi \text{ is a partial homomorphism} \\
\text{and } [(a,i,m)(b,j,n)]\psi &= (a,i,m)\psi(b,j,n)\psi.
\end{aligned}$$

As before, we may assume that $1 \in F \cap \Lambda$ and $q_{11} = e_H$, the identity of the structure group $H$.

By theorem 1.17, $(a,i,m)\varphi = u_i a\omega v_m$ where $i \to u_i$ and $m \to v_m$ are mappings of $F$ into $G$ and $\Lambda$ into $G$ respectively and $\omega$ is a homomorphism of $H$ into $G$ such that $q_{mj}\omega = v_m u_j$ if $q_{mj} \neq 0$.

On the other hand, let $(e,i,1)\psi = \alpha_i$, $(e,1,m)\psi = \beta_m$ and $(a,1,1)\psi = \gamma_a$ where $i \to \alpha_i$, $m \to \beta_m$ and $a \to \gamma_a$ are mappings of $F$ into $Y$, $\Lambda$ into $Y$ and $H$ into $Y$ respectively.

Now since $(a,1,1)(b,1,1) = (ab,1,1)$.

So, $(a,1,1)\psi(b,1,1)\psi = (ab,1,1)\psi$

$\Rightarrow \gamma_a \cdot \gamma_b = \gamma_{ab}$ for all $a, b \in H$ since $a$ and $b$ are arbitrary elements.

And since $(a,i,m) = (e,i,1)(a,1,1)(e,1,m)$.

Therefore

$$
\begin{aligned}
(a,i,m)\psi &= (e,i,1)\psi(a,1,1)\psi(e,1,m)\psi \\
&= \alpha_i\gamma_a\beta_m.
\end{aligned}
$$

and consequently, $\alpha_i\gamma_a\beta_m \in Y$ for all $i \in F$, $a \in H$ and $m \in \Lambda$.

Also since if $q_{mi} \neq 0$ $(e,1,m)(e,i,1) = (q_{mi},1,1)$.

Thus if $q_{mi} \neq 0, (e,1,m)\psi(e,i,1)\psi = (q_{mi},1,1)\psi$

$\Rightarrow$ if $q_{mi} \neq 0$, $\beta_m\alpha_i = \gamma_{q_{mi}}$.

Therefore, we conclude that,

$$(a,i,m)\pi = (u_i a\omega v_m, \alpha_i\gamma_a\beta_m).$$

Finally, we want to determine the multiplication "$o$" of $V = S \cup T^*$ in this case where $(T^*)\pi \subseteq \bar{S}\backslash S$.

Let $(a,i,m),(b,j,n) \in T^*$ and $((n,k),(g,p)),((r,s),(h,q)) \in S$. So, if $q_{mj} \neq 0$, $(a,i,m)o(b,j,n) = (a,i,m)(b,j,n)$

but if $q_{mj} = 0$,

$$
\begin{aligned}
(a,i,m)o(b,j,n) &= (a,i,m)\pi \cdot (b,j,n)\pi \\
&= (u_i a\omega v_m, \alpha_i\gamma_a\beta_m) \cdot (u_j b\omega v_n, \alpha_j\gamma_b\beta_n) \\
&= ((0,0),(u_i a\omega v_m u_j b\omega v_n, k_0))
\end{aligned}
$$

where $(p)\alpha_i\gamma_a\beta_m\alpha_j\gamma_b\beta_n = (l_0)\gamma_b\beta_n = k_0 \quad \forall p \in P$ where $q\beta_m\alpha_j = l_0 \quad \forall q \in P$ and this is because we know that if $q_{mj} = 0$ then $\beta_m\alpha_j$ is a constant mapping (note, use Theorem 2.7),

$$
\begin{aligned}
(a,i,m)o((n,k),(g,p)) &= (a,i,m)\pi \cdot ((n,k),(g,p)) \\
&= (u_i a\omega v_m, \alpha_i\gamma_a\beta_m) \cdot ((n,k),(g,p)) \\
&= ((n,k),(((u_i a\omega v_m)\theta^n)g,p)), \\
((n,k),(g,p))o(a,i,m) &= ((n,k),(g,p)) \cdot (a,i,m)\pi \\
&= ((n,k),(g,p)) \cdot (u_i a\omega v_m, \alpha_i\gamma_a\beta_m) \\
&= \begin{cases} ((n,k),(g((u_i a\omega v_m)\theta^k), p((u_i a\omega v_m)\theta^{k-1}\gamma))) & \text{if } k > 0 \\ ((n,0),(g(u_i a\omega v_m),(p)\alpha_i\gamma_a\beta_m)) & \text{if } k = 0 \end{cases}
\end{aligned}
$$

and $((n,k),(g,p))o((r,s),(h,q) = ((n,k),(g,p))((r,s),(h,q))$ where "$\cdot$" denotes the multiplication of $\bar{S}$ and juxtaposition denotes the multiplications of $T$ and $S$.

Next, we wish to show that $V = S \cup T^*$ under the multiplication given by (1)–(4) is an extension of $S$ by $T$. We will utilize theorem 1.15. Let $i \to u_i$, $m \to v_m$, $w$, $i \to \alpha_i$, $m \to \beta_m$ and $a \to \gamma_a$ be as in the statement of the theorem.

Define $\pi : T^* \to \bar{S}$ such that $(a,i,m)\pi = (u_i a\omega v_m, \alpha_i\gamma_a\beta_m)$. We will show that $\pi$ defines a partial homomorphism of $T^*$ into $\bar{S}$.

Let $(a,i,m),(b,j,n) \in T^*$ and suppose that $q_{mj} \neq 0$, by theorem 1.17, it only needs to show that $[(a,i,m)(b,j,n)]\psi = (a,i,m)\psi(b,j,n)\psi$.

$$[(a,i,m)(b,j,n)]\psi \;=\; (aq_{mj}b,i,n)\psi$$

$$= \alpha_i\gamma_{aq_{mj}b}\beta_n$$

$$= \alpha_i\gamma_a\gamma_{q_{mj}}\gamma_b\ \beta_n$$

$$= \alpha_i\gamma_a\beta_m\alpha_j\gamma_b\beta_n$$

$$= (a,i,m)\psi\cdot(b,j,n)\psi.$$

So, $\pi$ is indeed a partial homomorphism of $T^*$ into $\bar{S}$. Therefore, by theorem 1.14, $\pi$ determines an extension $(\bar{V},o)$ of $\bar{S}$ by $T$ as follows

$$(a,i,m)o(b,j,n) = \begin{cases} (a,i,m)(b,j,n) & \text{if } q_{mj} \neq 0 \\ (a,i,m)\pi\cdot(b,j,n)\pi & \text{if } q_{mj} = 0, \end{cases}$$

$\bar{s}o(a,i,m) = \bar{s}\cdot(a,i,m)\pi$, $(a,i,m)o\bar{s} = (a,i,m)\pi\cdot\bar{s}$ and $\bar{s}o\bar{t} = \bar{s}\cdot\bar{t}$ where juxtaposition denotes the multiplication of $T$ and "." denotes the multiplication of $\bar{S}$.

Hence, using theorem 1.15, $(V,o)$ is an extension of $S$ by $T$ if and only if $(a,i,m)o(b,j,n)\in S$ if $q_{mj}=0$. Using the condition, $\beta_m\alpha_j$ is a constant mapping if $q_{mj}=0$, $(a,i,m)o(b,j,n) = (a,i,m)\pi\cdot(b,j,n)\pi = ((0,0),(u_ia\omega v_m u_j b\omega v_n, k_0))$ where $(p)\alpha_i\gamma_a\beta_m\alpha_j\gamma_b\beta_n = k_0 \quad \forall p\in P$ where $q\beta_m\alpha_j = l_0 \quad \forall q\in P$. So, $(V,o)$ is an extension of $S$ by $T$. We have already shown $(a,i,m)o(b,j,n)$ is given by the second part of (1). It is easily checked that $((n,k),(g,p))o(a,i,m)$ and $(a,i,m)o((n,k),(g,p))$ are given by (2) and (3) respectively. The first part of (1) and (4) are valid since $(V,o)$ is an extension of $S$ by $T$.

**Theorem 3.3** *A mapping $\varphi$ of an E-bisimple semigroup $S=(G,P,K,\theta,\gamma)$ into* *a group $H$ is a homomorphism if and only if*

$$((n,k),(g,p))\varphi = t^n g\eta t^{-k}$$

*where $t \in H$ and $\eta$ is a homomorphism of $G$ into $H$ such that for all $g \in G$, $g\eta = t(g\theta)\eta t^{-1}$.*

Proof: Let $S$ be an $E$-bisimple semigroup and let $H$ be a group. Let $\varphi$ be a homomorphism of $S$ into $H$. Since, by lemma 1.1, $\varphi$ maps idempotent elements of $S$ into idempotent elements of $H$ and since the only idempotent element of $H$ is $e_H$, then $((0,0),(e_G,p))\varphi = e_H$ for all $p \in P$ and $((n,n),(e_G,q))\varphi = e_H$ for all $n \in N$ and $q \in K$.

Choose fixed elements $p_0 \in P$ and $q_0 \in K$ and let $((1,0),(e_G,p_0))\varphi = t$ and $((0,1),(e_G,q_0))\varphi = v$ where $t,v \in H$. We claim that $((n,0),(e_G,p_0))\varphi = t^n$ for all $n \in N$. To show this, we are going to use mathematical induction. Since

$$\begin{aligned}
((2,0),(e_G,p_0)) &= ((1,0),(e_G,p_0))((1,0),(e_G,p_0)) \\
\Rightarrow ((2,0),(e_G,p_0))\varphi &= ((1,0),(e_G,p_0))\varphi((1,0),(e_G,p_0))\varphi \\
&= t\cdot t = t^2.
\end{aligned}$$

So, it is true for $n = 2$.

Assume that it is true for $n = i$ i.e., $((i,0),(e_G,p_0))\varphi = t^i$. Then we want to show that it is true for $n = i+1$. Since

$$((i+1,0),(e_G,p_0)) = ((i,0)(e_G,p_0))((1,0),(e_G,p_0))$$

$$\Rightarrow ((i+1,0),(e_G,p_0))\varphi = ((i,0),(e_G,p_0))\varphi((1,0),(e_G,p_0))\varphi$$
$$= t^i \cdot t = t^{i+1}.$$

So, $((n,0),(e_G,p_0))\varphi = t^n$ for all $n \in N$. Similarly, we can easily show that $((0,k),(e_G,q_0))\varphi = v^k$ for all $k \in N$. Moreover for any $p \in P$ and $n \in N$

$$\begin{aligned}
((n,0),(e_G,p))\varphi &= ((n,0),(e_G,p))\varphi \cdot e_H \\
&= ((n,0),(e_G,p))\varphi((0,0),(e_G,p_0))\varphi \\
&= [((n,0),(e_G,p))((0,0),(e_G,p_0))]\varphi \\
&= ((n,0),(e_G,p_0))\varphi \\
&= t^n
\end{aligned}$$

and for any $q \in K$ and $k \in N$

$$\begin{aligned}
((0,k),(e_G,q))\varphi &= ((0,k),(e_G,q)) \cdot e_H \\
&= ((0,k),(e_G,q))\varphi((k,k),(e_G,q_0))\varphi \\
&= [((0,k),(e_G,q))((k,k),(e_G,q_0))]\varphi \\
&= ((0,k),(e_G,q_0))\varphi \\
&= v^k.
\end{aligned}$$

Also, since

$$\begin{aligned}
tv &= ((1,0),(e_G,p_0))\varphi((0,1),(e_G,q_0))\varphi \\
&= [((1,0),(e_G,p_0))((0,1),(e_G,q_0))]\varphi \\
&= ((1,1),(e_G,q_0))\varphi \\
&= e_H.
\end{aligned}$$

Therefore, $v = t^{-1}$. So, for any $q \in K$ and $k \in N$, $((0,k),(e_G,q))\varphi = t^{-k}$.

Next, define a mapping $\eta$ from $G$ into $H$ such that $((0,0),(g,l_0))\varphi = g\eta$ where $l_0 \in P$ is a fixed element. But since

$$((0,0),(g,p))((0,0),(e_G,l_0)) = ((0,0),(g,l_0)) \quad \text{where } p \in P.$$

Then

$$\begin{aligned} ((0,0),(g,p))\varphi((0,0),(e_G,l_0))\varphi &= ((0,0),(g,l_0))\varphi \\ \Rightarrow ((0,0),(g,p))\varphi \cdot e_H &= g\eta \end{aligned}$$

Thus, for any $p \in P$, $((0,0),(g,p))\varphi = g\eta$.

Also, since for all $g_1, g_2 \in G$,

$$\begin{aligned} (g_1g_2)\eta &= ((0,0),(g_1g_2,l_0))\varphi \\ &= [((0,0),(g_1,l_0))((0,0),(g_2,l_0))]\varphi \\ &= ((0,0),(g_1,l_0))\varphi((0,0),(g_2,l_0))\varphi \\ &= g_1\eta \cdot g_2\eta. \end{aligned}$$

Therefore, $\eta$ is a homomorphism from $G$ into $H$.

Now since $((n,k),(g,q)) = ((n,0),(e_G,z))((0,0),(g,z))((0,k),(e_G,q))$ where $z \in P$ is any element. Hence

$$\begin{aligned} ((n,k),(g,q))\varphi &= ((n,0),(e_G,z))\varphi((0,0),(g,z))\varphi((0,k),(e_G,q))\varphi \\ &= t^n g\eta t^{-k}. \end{aligned}$$

Finally, since

$$\begin{aligned}((0,1),(e_G,p))((0,0),(g,q)) &= ((0,1),(g\theta,p(g\gamma)))\\ \Rightarrow ((0,1),(e_G,p))\varphi((0,0),(g,q))\varphi &= ((0,1),(g\theta,p(g\gamma)))\varphi\\ t^{-1}g\eta &= (g\theta)\eta t^{-1}.\end{aligned}$$

Multiplying both sides by $t$ from the left we get,

$$g\eta = t(g\theta)\eta t^{-1}.$$

Conversely, let us suppose that $\varphi$ is a mapping from $S$ into $H$ defined as $((n,k),(g,p))\varphi = t^n g\eta t^{-k}$ where $t \in H$ and $\eta$ is a homomorphism of $G$ into $H$ such that for all $g \in G$, $\quad g\eta = t(g\theta)\eta t^{-1}$.

First, we claim that $g\eta = t^m(g\theta^m)\eta t^{-m}$ for all $m \in N$ and $g \in G$. To show that, we will use mathematical induction. For $m = 1$, it is given that for all $g \in G, g\eta = t(g\theta)\eta t^{-1}$. Next, assume that it is true for $m = j$ i.e., for all $g \in G, g\eta = t^j(g\theta^j)\eta t^{-j}$. Then, we want to show that it is true for $m = j + 1$. Since $g\theta^j \in G$, so, $(g\theta^j)\eta = t((g\theta^j)\theta)\eta t^{-1} = t(g\theta^{j+1})\eta t^{-1}$. Therefore, $g\eta = t^j(t(g\theta^{j+1})\eta t^{-1})t^{-j} = t^{j+1}(g\theta^{j+1})\eta t^{-(j+1)}$ for all $g \in G$. Hence for all $g \in G$ and $m \in N \quad g\eta = t^m(g\theta^m)\eta t^{-m} \qquad (*)$.

Now we want to show that $\varphi$ is a homomorphism of $S$ into $H$.

For $((n,k),(g,p)),((r,s),(f,q)) \in S$, first let $r > k$

$$\begin{aligned}[((n,k),(g,p))((r,s),(f,q))]\varphi &= ((n+r-k,s),(g\theta^{r-k}f,q))\varphi\\ &= t^{n+r-k}(g\theta^{r-k}f)\eta t^{-s}\end{aligned}$$

$$= t^n \cdot t^{r-k}(g\theta^{r-k})\eta f\eta t^{-s}$$

$$= t^n g\eta t^{r-k} f\eta t^{-s} \qquad \text{(from (*))}$$

$$= t^n g\eta t^{-k} t^r f\eta t^{-s}$$

$$= ((n,k),(g,p))\varphi((r,s),(f,q))\varphi$$

Next, let $k > r$

$$\begin{aligned}
[((n,k),(g,p))((r,s),(f,q))]\varphi &= ((n,k+s-r)(g(f\theta^{k-r}),p(f\theta^{k-r-1}\gamma)))\varphi \\
&= t^n(g(f\theta^{k-r}))\eta t^{-(k+s-r)} \\
&= t^n g\eta(f\theta^{k-r})\eta t^{-(k-r)}t^{-s} \\
&= t^n g\eta t^{-(k-r)} f\eta t^{-s} \qquad \text{( from (*))} \\
&= t^n g\eta t^{-k} t^r f\eta t^{-s} \\
&= ((n,k),(g,p))\varphi((r,s),(f,q))\varphi
\end{aligned}$$

For $k = r$, the case is very clear. Therefore, $\varphi$ is a homomorphism of $S$ into $H$.

**Theorem 3.4** *Let $S = M^0(H,I,I,\Delta)$, where $I$ is a finite set, be a Brandt semigroup and let $T^* = (G,P,K,\theta,\gamma)$ be an E-bisimple semigroup. Let $V$ be an extension of $S$ by $T$. Then, there exists a homomorphism $W : T^* \to G_X$, the full symmetric group on some $r$ element subset $X$ of $I$ such that $AW = W_A$. This homomorphism is explicitly given by Theorem 3.3. For each $A \in T^*$, there exists a mapping $\psi_A$ of $X$ into the group $H$ such that*

$$(i\psi_A)(iW_A\psi_B) = i\psi_{AB} \quad \textit{for all } i \in X.$$

*The multiplication of $V$ is determined as follows*

$$AoB = AB \tag{9}$$

$$(a,i,m)oA = \begin{cases} (a(m\psi_A), i, mW_A) & \text{if } m \in X \\ 0, \ \text{the zero of } S, & \text{if } m \notin X \end{cases} \tag{10}$$

$$Ao(a,i,m) = \begin{cases} ((iW_A^{-1}\psi_A)a, iW_A^{-1}, m) & \text{if } i \in X \\ 0 & \text{if } i \notin X \end{cases} \tag{11}$$

$$Ao0 = 0oA = 0. \tag{12}$$

*Conversely, let $S$ be a Brandt semigroup and let $T^*$ be an $E$-bisimple semigroup such that $T \cap S = \Box$. If we are given the mappings $W$ and $\psi_A$ described above and define products "o" in the set $V = S \cup T^*$ by (5) - (8), then $V$ is an extension of $S$ by $T$.*

<u>Proof</u>: Let $V$ be an extension of $S$ by $T$. Then, by theorem 1.16, there exists a partial homomorphism $W$ of $T^*$ into $\mathcal{I}_I$, the full symmetric inverse semigroup on $I$ such that $AW = W_A$ and for each $A \in T^*$, there exists a mapping $\psi_A$ of $s_A$, the domain of $W_A$, into the group $H$ such that

$$(i\psi_A)(iW_A\psi_B) = i\psi_{AB} \quad \text{for all } i \in s_{AB}, \text{ the domain of } W_{AB}$$

and the multiplication of $V$ is determined as in the statement of that theorem. But since for any $A, B \in T^*$, $AB$ never equal to zero, $W$ is a homomorphism of $T^*$ into $\mathcal{I}_I$. We claim that $T^*W$ is a group. Since, by lemma 1.3, a homomorphism maps $\mathcal{D}$-classes into $\mathcal{D}$-classes and since $T^*$ is a single $\mathcal{D}$-class, $T^*W$ is contained in a single $\mathcal{D}$-class of $\mathcal{I}_I$. So, $T^*W$ is a bisimple semigroup. Moreover, since every bisimple semigroup is a simple semigroup, $T^*W$ is a simple semigroup. But since $I$ is a finite set, $\mathcal{I}_I$ is also a finite set which implies that $T^*W$ is a finite set too. So, $T^*W$ is a completely simple semigroup since it is a finite simple semigroup. Furthermore, since $T^*W \subseteq \mathcal{I}_I$, then $T^*W$ is an inverse semigroup which implies that $T^*W$ is regular, and any two idempotent elements of $T^*W$ commute with each other by theorem 1.2. Now, let $T^*W$ be given the Rees representation $M(H_1, F, \Lambda, Q)$ and consider that the sandwich matrix $Q$ to be normalized so that all the elements in the first row and in the first column are the identity element $e_{H_1}$ of the structure group $H_1$ by Lemma 1.8 and $q_{11} = e_H$ where $1 \in F \cap \Lambda$.

So, for any idempotent element $(q_{mi}^{-1}, i, m) \in T^*W$, we have

$$\begin{aligned} (e_{H_1}, 1, 1)(q_{mi}^{-1}, i, m) &= (q_{mi}^{-1}, i, m)(e_{H_1}, 1, 1) \\ \Rightarrow (q_{mi}^{-1}, 1, m) &= (q_{mi}^{-1}, i, 1) \\ \Rightarrow i = 1 \text{ and } m &= 1. \end{aligned}$$

Therefore, $F = \{1\}$ and $\Lambda = \{1\}$. Hence, any element of $T^*W$ can be represented as $(a, 1, 1)$ where $a \in H_1$. Moreover, since

$$(e_{H_1}, 1, 1)(a, 1, 1) = (a, 1, 1)(e_{H_1}, 1, 1) = (a, 1, 1)$$

for all $(a, 1, 1) \in T^*W$, then $(e_{H_1}, 1, 1)$ is an identity element of $T^*W$. Furthermore, since for any element $(a, 1, 1) \in T^*W$, there exists an element $(a^{-1}, 1, 1) \in T^*W$ such that

$$(a, 1, 1)(a^{-1}, 1, 1) = (a^{-1}, 1, 1)(a, 1, 1) = (e_{H_1}, 1, 1)$$

the inverse property is satisfied. Therefore, $T^{*}W$ is a group.

We now turn to the proper consideration that the group $T^{*}W$ is contained in $\mathcal{I}_I$ and let us suppose that $W_E$ be the identity element of $T^{*}W$. Since the set of idempotent elements of $\mathcal{I}_I$, $E(\mathcal{I}_I) = \{\alpha \in \mathcal{I}_I | \alpha$ is the identity mapping on some subset $X \subseteq I\}$, and since $W_E W_E = W_E$. So, $W_E \in E(\mathcal{I}_I)$. Thus, there exists $X \subseteq I$ such that $W_E$ is a mapping of $X$ onto $X$ and $xW_E = x$ for all $x \in X$.

Now since $T^{*}W$ is group with identity element $W_E$. So, for any element $W_A \in T^{*}W$, there exists $\alpha \in T^{*}W$ such that

$$W_A\alpha = \alpha W_A = W_E$$

Hence,

$$W_A\alpha W_A = W_E W_A = W_A$$

&

$$\alpha W_A \alpha = \alpha W_E = \alpha.$$

Therefore, since $\mathcal{I}_I$ is an inverse semigroup, $\alpha = W_A^{-1}$ where $W_A^{-1} : t_A \to s_A$ since $W_A : s_A \to t_A$, where $s_A$ is the domain of $W_A$ and $t_A$ is the range of $W_A$. Thus

$$W_A W_A^{-1} = W_A^{-1} W_A = W_E.$$

Hence

$$s_A = \text{Dom } W_A W_A^{-1} = \text{Dom } W_E = X,$$

and

$$t_A = \text{Range } W_A^{-1}W_A = \text{Range } W_E = X.$$

Therefore, for any $W_A \in T^{*}W$, $s_A = t_A = X$. Therefore, $T^{*}W$ is a subgroup of $G_X$, the full symmetric group on the subset $X$ of $I$.

The converse is a consequence of theorem 1.16.

<u>Remark</u>: Theorem 3.4 remains true with the same proof even if we replace the $E$-bisimple semigroup by any simple semigroup.

# REFERENCES


1. Baird, G.R. *On a sublattice of the lattice of congruences on a simple regular-semigroup*, J. Austral. Math. Soc. 13(1972), 461-471.

2. Baird, G.R., *Congruences on simple regular-semigroups*, J. Australian Math. Soc. 14(1972), 155-167.

3. Bonzine, C. and Cherubini A., *Modularity of the lattice of congruences on a regular $\omega$-semigroup*, Proc. Edinburgh Math. Soc. 33(1990), 405-417.

4. Bonzini, C. and Cherubini, A., *Semimodularity of the lattice of congruences on regular $\omega$-semigroups*, nh.math. 109(1990), 205-219.

5. Bonzini, C. and Cherubini, A., *Conditions similar to modularity on the congruence lattice of a regular $\omega$-semigroup*, Proc. of the International symposium on the Semigroup Theory, Kyoto, (1990), 41-51.

6. Clifford, A.H., *Extensions of Semigroups*, Trans. Amer. Math. Soc. 68(1950), 165-173.

7. Clifford, A.H. and G.B. Preston, *The algebraic theory of semigroups*, Math. Surveys No. 7, Amer. Math. Soc., Providence, R.I., 1961.

8. Howie, J.M., *An Introduction to Semigroup Theory*, Academic Press Inc. (London) Ltd., 1976.

9. Munn, W.D., *Regular $\omega$-semigroups*, Glasgow math. J. 9(1968), 46-66.

10. Munn, W.D., *The lattice of congruences on a bisimple $\omega$-semigroup*, Proc. Roy. Soc. Edinburgh A67(1965/67), 175-184.

11. Munn, W.D. and N.R. Reilly, *Congruences on a bisimple $\omega$-semigroup*, Proc. Glasgow Math. Assoc. 7(1966), 184-192.

12. Petrich, M., *Congruences on simple $\omega$-semigroups*, Glasgow Math. J. 20(1979), 87-101.

13. Reilly, N.R., *Bisimple $\omega$-semigroups*, Proc. Glasgow Math. Assoc. 7)1966), 160-169.

14. Schlubich, H.E., *Semimodularity and bisimple $\omega$-semigroups*, Proc. Edinburgh Math. Soc. 17(1970), 79-81.

15. Warne, R.J., *Extensions of completely 0-simple semigroups by completely 0-simple semigroups*, Proc. Amer. Math. Soc. 17(1966), 524-526.

16. Warne, R.J., *Extensions of Brandt semigroups and applications*, Illinois J. Math. 10(1966), 652-660.

17. Warne, R.J., *A class of bisimple inverse semigroups*, Pacific J. Math. 18(1966), 563-577.

18. Warne, R.J., *Extensions of Brandt semigroups*, B.A.M.S. 72(1966), 683-684.

19. Warne, R.J., *Extensions of I-bisimple semigroups*, Canad. J. Math. 19(1967), 419-426; 20(1968), 511-512.

20. Warne, R.J., *Bisimple inverse semigroups mod groups*, Duke Math. J. 34(1967), 787-812.

21. Warne, R.J., *I-bisimple semigroups*, Trans. Amer. Math. Soc. 150(1968), 367-386.

22. Warne, R.J., *Extensions of $\omega^n$ I-bisimple semigroups*, Math. Japon. 13(1968), 105-121.

23. Warne, R.J., *Congruences on $\omega^n$-bisimple semigroups*, Jour Austral. Math. Soc. 9(1969), 57-274.

24. Warne, R.J., *I-regular semigroups*, Math. Japon. 15(1970), 91-100.

25. Warne, R.J., *Some properties of simple I-regular semigroups*, Composito Math. 22(1970), 181-195.

26. Warne, R.J., *$\omega^n$ I-bisimple semigroups*, Acta Math. Acad. Sci, Hung. 21(1970), 121-150.

27. Warne, R.J., *E-bisimple semigroups*, Journal of Natural Sciences and Math. (W. Pakistan) 11(1971), 51-81.

28. Warne, R.J., *Embedding of regular semigroups in wreath products*, J. Pure Appl. Algebra 29(1983), 177-207.